

POLYMORPHISM CLONES OF HOMOGENEOUS STRUCTURES

UNIVERSAL HOMOGENEOUS POLYMORPHISMS AND AUTOMATIC HOMEOMORPHICITY

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ABSTRACT. Every clone of functions comes naturally equipped with a topology—the topology of pointwise convergence. A clone \mathcal{C} is said to have automatic homeomorphicity with respect to a class \mathcal{C} of clones, if every clone-isomorphism of \mathcal{C} to a member of \mathcal{C} is already a homeomorphism (with respect to the topology of pointwise convergence). In this paper we study automatic homeomorphicity-properties for polymorphism clones of countable homogeneous relational structures. To this end we introduce and utilize universal homogeneous polymorphisms. Our results extend and generalize previous results by Bodirsky, Pinsker, and Pongrácz.

1. INTRODUCTION

A relational structure is called homogeneous if every isomorphism between finite substructures extends to an automorphism. Homogeneous structures play an important role in model theory because of their close relation to structures whose elementary theory admits quantifier elimination. Also, homogeneous structures form a major source of ω -categorical structures.

A clone is a set of finitary functions on a given base set that contains all projections and that is closed with respect to composition.

Until recently, the concepts of clones and of homogeneous structures seldom were mentioned in one sentence, because they inhabited different branches of mathematics—general algebra from the one hand (cf. [18,37,42]) and model theory, combinatorics and group theory, on the other hand (cf. [9,21,29]). However, they became linked by the theory of constraint satisfaction problems (cf. [5,7,8]). In particular, homogeneous structures appear as templates of constraint satisfaction problems and their clone of polymorphisms largely determines the complexity of named problem. To be more precise, it has been shown by Bodirsky and Pinsker [5] that the complexity of a CSP with ω -categorical template is determined by the underlying abstract clone of the polymorphism clone of its template, together with the topology of pointwise convergence. The authors of this paper asked, under which conditions the complexity is already determined by the underlying *abstract* clone alone (i.e., without the topology). Certainly, a sufficient condition is that the canonical topology of the polymorphism clone of the template can be reconstructed from its underlying abstract clone. First steps to find reasonably general conditions were undertaken by Bodirsky, Pinsker and Pongrácz in [6]. Our paper is build on their findings.

What is meant by “reconstructing the canonical topology of a clone”? There are several ways to give concrete meaning to the phrase: For a class \mathcal{K} of clones and a clone $\mathcal{C} \in \mathcal{K}$ we may say that

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- (1) \mathfrak{C} has *reconstruction with respect to \mathcal{K}* if whenever \mathfrak{C} is isomorphic to some clone $\mathfrak{D} \in \mathcal{K}$ (as an abstract clone), then there exists already an isomorphism between \mathfrak{C} and \mathfrak{D} that is a homeomorphism (with respect to the canonical topologies of \mathfrak{C} and \mathfrak{D} , respectively), or
- (2) \mathfrak{C} has *automatic homeomorphicity with respect to \mathcal{K}* if whenever \mathfrak{C} is isomorphic to some clone $\mathfrak{D} \in \mathcal{K}$ (as an abstract clone), then *every* isomorphism between \mathfrak{C} and \mathfrak{D} is a homeomorphism.

In this paper we are going to study the second (stronger) option. Note that automatic homeomorphicity is already a non-trivial concept if the class \mathcal{K} consists only of \mathfrak{C} . In this case it says that every automorphism of \mathfrak{C} is an autohomeomorphism. As a matter of fact, \mathfrak{C} has automatic homeomorphicity if and only if it has reconstruction and if every automorphism of \mathfrak{C} is a homeomorphism.

It should be mentioned that our approach to automatic homeomorphicity is not that of a craftsman but of an engineer. That is, our goal is not, for every given homogeneous structure in question to find the shortest and most elegant proof that its polymorphism clone has automatic homeomorphicity. Rather it is our ambition to find methods as general as possible to show automatic homeomorphicity of the polymorphism clones of whole classes of structures at once. We do so by refining and industrializing the gate techniques that were introduced in [6]. In particular:

- (1) we introduce the notion of strong gate coverings,
- (2) we show, how strong gate coverings can be used for showing automatic homeomorphicity of clones,
- (3) we introduce the notion of universal homogeneous polymorphisms,
- (4) we show that the existence of universal homogeneous polymorphisms of all finite arities for a relational structure implies that its polymorphism clone has a strong gate covering,
- (5) we characterize all homogeneous structures that possess universal homogeneous polymorphisms of all finite arities by a property of their age,

Thus we end up with a sufficient condition for the existence of strong gate coverings for polymorphism clones of homogeneous structures. In particular, we show the existence of strong gate coverings for the polymorphism clones of the following structures:

- free homogeneous structures whose age has the homo-amalgamation property and is closed with respect to finite products,
- the generic poset (with reflexive order relation),

Moreover, we show that the following structures do not have universal homogeneous polymorphisms of any arity ≥ 2 :

- the rational Urysohn space,
- the rationals (\mathbb{Q}, \leq) .

The paper concludes with new criteria for the automatic homeomorphicity of clones. In particular we show that the polymorphism clone of a free homogeneous structure \mathbf{U} has automatic homeomorphicity if

- (i) $\text{Age}(\mathbf{U})$ has the homo-amalgamation property,
- (ii) $\text{Age}(\mathbf{U})$ is closed with respect to finite products,
- (iii) all constant functions on U are endomorphisms of \mathbf{U} .

Moreover, we show that in the above criterion condition (iii) can be replaced by the following two conditions:

- (iii.a) $\text{Aut}(\mathbf{U})$ acts transitively on U ,
- (iii.b) $\overline{\text{Aut}}(\mathbf{U})$ has automatic homeomorphicity.

Finally, we present a result on automatic homeomorphicity for one non-free homogeneous structure. In particular we shown that the polymorphism clone of the generic poset (with reflexive order-relation) has automatic homeomorphicity.

Some words about the techniques employed by us. For the part about universal homogeneous polymorphisms we use *axiomatic Fraïssé theory*. This is a version of Fraïssé theory, introduced by Droste and Göbel in [14], that completely abstracts from structures. It is formalized in the language of category theory and encompasses model theoretic Fraïssé-theory (including, e.g., Hrushovski's construction and Solecki's projective Fraïssé-limits) and, what is known in model theory as back and forth techniques. The theory has meanwhile been applied, developed, and extended in several works, including [10,25–27,32–34,39]. We build upon the results from [33] on universal homogeneous objects in comma-categories and extend them, in order to obtain our characterization of the existence of universal homogeneous polymorphisms for homogeneous structures.

Another important tool in our research has been a topological version of Birkhoff's theorem due to Bodirsky and Pinsker [5] in a rather surprising combination with results about polymorphism homogeneous structures and retracts of Fraïssé-limits (cf. [33,35]).

2. PRELIMINARIES

2.1. Clones. Let A be a set, For $n \in \mathbb{N} \setminus \{0\}$ we define

$$\mathfrak{O}_A^{(n)} := \{f \mid f: A^n \rightarrow A\}, \text{ and}$$

$$\mathfrak{O}_A := \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathfrak{O}_A^{(n)}.$$

In general, for a set $C \subseteq \mathfrak{O}_A$ we will write $C^{(n)}$ for the set of all n -ary functions from C .

We distinguish certain functions in \mathfrak{O}_A —the *projections*: For $n \in \mathbb{N} \setminus \{0\}$, and for $i \in \{1, \dots, n\}$ the projection $e_i^n \in \mathfrak{O}_A^{(n)}$ is defined by

$$e_i^n: (x_1, \dots, x_n) \mapsto x_i.$$

Further we define the set of all projections on A :

$$\mathfrak{I}_A := \{e_i^n \mid n \in \mathbb{N} \setminus \{0\}, i \in \{1, \dots, n\}\}.$$

For all $n, m \in \mathbb{N} \setminus \{0\}$, whenever $f \in \mathfrak{O}_A^{(n)}$, and $g_1, \dots, g_n \in \mathfrak{O}_A^{(m)}$, then the *composition* $f \circ \langle g_1, \dots, g_n \rangle \in \mathfrak{O}_A^{(m)}$ is defined according to

$$f \circ \langle g_1, \dots, g_n \rangle: (x_1, \dots, x_m) \mapsto f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)).$$

Definition 2.1. A set $\mathfrak{C} \subseteq \mathfrak{O}_A$ is called *clone* on A if

- (1) $\mathfrak{I}_A \subseteq \mathfrak{C}$,
- (2) \mathfrak{C} is closed with respect to composition.

Clearly, both, \mathfrak{O}_A and \mathfrak{I}_A are clones. If \mathfrak{C} and \mathfrak{D} are clones on A , and if $\mathfrak{C} \subseteq \mathfrak{D}$, then we call \mathfrak{C} a *subclone* of \mathfrak{D} , and we denote this fact by $\mathfrak{C} \leq \mathfrak{D}$.

Definition 2.2. Let A, B be sets and let $\mathfrak{C} \leq \mathfrak{O}_A$, $\mathfrak{D} \leq \mathfrak{O}_B$. A function $h: \mathfrak{C} \rightarrow \mathfrak{D}$ is called a *clone-homomorphism* if

- (1) for all $n \in \mathbb{N} \setminus \{0\}$ we have $h(\mathfrak{C}^{(n)}) \subseteq \mathfrak{D}^{(n)}$,
- (2) for all $n \in \mathbb{N} \setminus \{0\}$ and for all $i \in \{1, \dots, n\}$ we have $h(e_i^n) = e_i^n$,

(3) for all $n, m \in \mathbb{N} \setminus \{0\}$, for all $f \in \mathfrak{C}^{(n)}$, and for all $g_1, \dots, g_n \in \mathfrak{C}^{(m)}$ we have

$$h(f \circ \langle g_1, \dots, g_n \rangle) = h(f) \circ \langle h(g_1), \dots, h(g_n) \rangle.$$

A bijective clone-homomorphism will be called *clone-isomorphism*.

2.2. The Tychonoff topology on clones. Let U be a set and let $n \in \mathbb{N} \setminus \{0\}$. For every finite subset M of U^n and every $h: M \rightarrow U$ define

$$\Phi_{M,h} := \{f: U^n \rightarrow U \mid f|_M = h\}.$$

Then all the sets of this shape form the basis of a topology on $\mathfrak{D}_U^{(n)}$ —the Tychonoff topology (aka the topology of pointwise convergence; here U is considered to be equipped with the discrete topology). With this observation we may consider \mathfrak{D}_U as a topological sum

$$\mathfrak{D}_U = \bigsqcup_{n \in \mathbb{N} \setminus \{0\}} \mathfrak{D}_U^{(n)}.$$

Moreover, every clone $\mathfrak{C} \leq \mathfrak{D}_U$ may be equipped with the subspace topology with respect to the topology on \mathfrak{D}_U . This topology will be called the *canonical topology* of \mathfrak{C} . From now on, every clone will implicitly be considered to be equipped with its canonical topology.

Remark. Transformation monoids and permutation groups on U are subsets of $\mathfrak{D}_U^{(1)}$. Thus, they may be equipped with a subspace topology of $\mathfrak{D}_U^{(1)}$. As for clones, in the sequel we will consider every transformation monoid and every permutation group on U to be equipped with this topology, and we will call it the canonical topology of the respective transformation monoid or the permutation group.

If U is countably infinite, then, since the space $\mathfrak{D}_U^{(n)}$ is the countable power of a countable discrete space, the above given topology is completely metrizable by an ultrametric. In order to do so we consider U^n as an ω -indexed family $(\bar{u}_i)_{i < \omega}$. Now we consider the function

$$D_U^{(n)}: \mathfrak{D}_U^{(n)} \times \mathfrak{D}_U^{(n)} \rightarrow \omega^+$$

$$(f, g) \mapsto \begin{cases} \min\{i \in \omega \mid f(\bar{u}_i) \neq g(\bar{u}_i)\} & \text{if } f \neq g \\ \omega & \text{if } f = g. \end{cases}$$

Now, the mentioned ultrametric is given by

$$d_U^{(n)}(f, g) := \begin{cases} 2^{-D_U^{(n)}(f, g)} & \text{if } f \neq g \\ 0 & \text{if } f = g. \end{cases}$$

for $f, g \in \mathfrak{D}_U^{(n)}$.

Finally, the ultrametries $d_U^{(n)}$ may be combined to one ultrametric d_U on \mathfrak{D}_U according to

$$(1) \quad d_U(f, g) := \begin{cases} 1 & \text{if } f \in \mathfrak{D}_U^{(n)}, g \in \mathfrak{D}_U^{(m)}, n \neq m \\ d_U^{(n)}(f, g) & \text{if } f, g \in \mathfrak{D}_U^{(n)}. \end{cases}$$

At this point it is important to note that the metric space (\mathfrak{D}_U, d_U) is complete no matter how the enumerations of the $\mathfrak{D}_U^{(n)}$ for $n \in \mathbb{N} \setminus \{0\}$ are chosen. In particular, if we choose other enumerations of the $\mathfrak{D}_U^{(n)}$, and obtain an ultrametric, say, d'_U on \mathfrak{D}_U , then a sequence in \mathfrak{D}_U is going to be a Cauchy-sequence with respect to d_U if and only if it is a Cauchy-sequence with respect to d'_U . In the sequel, for any countable set U , we are going to consider \mathfrak{D}_U to be equipped with an ultrametric d_U , defined like in (1) through arbitrary enumerations of the $\mathfrak{D}_U^{(n)}$. Moreover, we will consider all subspaces of $\mathfrak{D}_U^{(n)}$ to

be equipped with the corresponding restriction of d_U , and we will (abusing notation) again denote the restriction by d_U .

2.3. Relational structures. A *relational signature* is a pair $\underline{\Sigma} = (\Sigma, \text{ar})$ where Σ is a set of *relational symbols* and $\text{ar}: \Sigma \rightarrow \mathbb{N} \setminus \{0\}$ assigns to each relational symbol its *arity*. The set of all n -ary relational symbols in Σ will be denoted by $\Sigma^{(n)}$.

A $\underline{\Sigma}$ -*structure* \mathbf{A} is a pair $(A, (\varrho^{\mathbf{A}})_{\varrho \in \Sigma})$, such that A is a set, and such that for each $\varrho \in \Sigma$ we have that $\varrho^{\mathbf{A}}$ is a relation of arity $\text{ar}(\varrho)$ on A . The set A will be called the *carrier* of \mathbf{A} and the relations $\varrho^{\mathbf{A}}$ will be called the *basic relations* of \mathbf{A} . If the signature $\underline{\Sigma}$ is of no importance, we will speak only about relational structures. The carriers of a $\underline{\Sigma}$ -structures $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ will usually be denoted by A, B, C, \dots , respectively. Moreover, the basic relations of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ will be denoted by $\varrho^{\mathbf{A}}, \varrho^{\mathbf{B}}, \varrho^{\mathbf{C}}, \dots$, respectively, for each $\varrho \in \Sigma$.

Let \mathbf{A} and \mathbf{B} be $\underline{\Sigma}$ -structures. A function $h: A \rightarrow B$ is called a *homomorphism* if for all $n \in \mathbb{N} \setminus \{0\}$, for all $\varrho \in \Sigma^{(n)}$ and for all $\bar{a} = (a_1, \dots, a_n) \in \varrho^{\mathbf{A}}$ we have that $h(\bar{a}) := (h(a_1), \dots, h(a_n)) \in \varrho^{\mathbf{B}}$. A function $h: A \rightarrow B$ is called *embedding* if h is injective and if for all $n \in \mathbb{N} \setminus \{0\}$, for all $\varrho \in \Sigma^{(n)}$ and for all $\bar{a} \in A^n$ we have

$$\bar{a} \in \varrho^{\mathbf{A}} \iff h(\bar{a}) \in \varrho^{\mathbf{B}}.$$

Surjective embeddings are called *isomorphisms*. As usual, isomorphisms of a relational structure \mathbf{A} onto itself are called *automorphisms*, and homomorphisms of \mathbf{A} to itself are called *endomorphisms*. The automorphism group and the endomorphism monoid of \mathbf{A} will be denoted by $\text{Aut}(\mathbf{A})$ and $\text{End}(\mathbf{A})$, respectively.

Whenever we write $h: \mathbf{A} \rightarrow \mathbf{B}$, we mean that h is a homomorphism from \mathbf{A} to \mathbf{B} . Moreover, with $h: \mathbf{A} \hookrightarrow \mathbf{B}$ we denote the fact that h is an embedding from \mathbf{A} into \mathbf{B} . Moreover, we write just $\mathbf{A} \hookrightarrow \mathbf{B}$ if there exists an embedding of \mathbf{A} into \mathbf{B} .

Let \mathbf{A} be a relational structure. For $n \in \mathbb{N} \setminus \{0\}$, a homomorphism $h: \mathbf{A}^n \rightarrow \mathbf{A}$ is called an *n -ary polymorphism* of \mathbf{A} . With $\text{Pol}^{(n)}(\mathbf{A})$ we will denote the set of all n -ary polymorphisms of \mathbf{A} . Moreover, we define

$$\text{Pol}(\mathbf{A}) := \bigcup_{n \in \mathbb{N} \setminus \{0\}} \text{Pol}^{(n)}(\mathbf{A}).$$

It is easy to see, that for every relational structure \mathbf{A} we have that $\text{Pol}(\mathbf{A})$ is a closed subclone of \mathfrak{D}_A —the polymorphism clone of \mathbf{A} . It is less obvious, that every closed subclone on \mathfrak{D}_A may be obtained as the polymorphism clone of a suitable relational structure on A (cf. [1, Lemma 3.1], [38, Theorem 1], [36, Theorem 4.1]).

2.4. Homogeneous structures. The age of a $\underline{\Sigma}$ -structure \mathbf{U} is the class of finite $\underline{\Sigma}$ -structures embeddable into \mathbf{U} . It will be denoted by $\text{Age}(\mathbf{U})$. A structure \mathbf{A} is called *younger* than \mathbf{U} if $\text{Age}(\mathbf{A}) \subseteq \text{Age}(\mathbf{U})$. According to a classical result by Fraïssé, a class \mathcal{C} of finite $\underline{\Sigma}$ -structures is the age of a countable $\underline{\Sigma}$ -structure if and only if

- (1) \mathcal{C} has the *hereditary property* (HP), i.e.

$$\forall \mathbf{A}, \mathbf{B} : (\mathbf{B} \in \mathcal{C}) \wedge (\mathbf{A} \hookrightarrow \mathbf{B}) \Rightarrow (\mathbf{A} \in \mathcal{C}),$$

- (2) \mathcal{C} has the *joint embedding property* (JEP), i.e.

$$\forall \mathbf{A}, \mathbf{B}, \in \mathcal{C} \exists \mathbf{C} \in \mathcal{C} : (\mathbf{A} \hookrightarrow \mathbf{C}) \wedge (\mathbf{B} \hookrightarrow \mathbf{C}),$$

- (3) up to isomorphism, \mathcal{C} contains only countably many structures.

Thus it is natural to call a class \mathcal{C} of finite $\underline{\Sigma}$ -structures with these three properties an *age*.

If \mathcal{C} is an age, then by $\overline{\mathcal{C}}$ we will denote the class of all countable structures whose age is contained in \mathcal{C} .

Definition 2.3. A countable $\underline{\Sigma}$ -structure \mathbf{A} is called *universal* if every structure from $\overline{\text{Age}(\mathbf{A})}$ can be embedded into \mathbf{A} . It is called *homogeneous* if for every $\mathbf{B} \in \text{Age}(\mathbf{A})$ and for all embeddings $\iota_1, \iota_2: \mathbf{B} \hookrightarrow \mathbf{A}$ there exists an automorphism h of \mathbf{A} such that $\iota_2 = h \circ \iota_1$.

Definition 2.4. Let \mathcal{C} be a class of $\underline{\Sigma}$ -structures. We say that \mathcal{C} has the *amalgamation property* (AP) if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ from \mathcal{C} and for all embeddings $f: \mathbf{A} \hookrightarrow \mathbf{B}$, $g: \mathbf{A} \hookrightarrow \mathbf{C}$, there exists $\mathbf{D} \in \mathcal{C}$ and embeddings $\hat{f}: \mathbf{C} \hookrightarrow \mathbf{D}$, $\hat{g}: \mathbf{B} \hookrightarrow \mathbf{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\hat{f}} & \mathbf{D} \\ \uparrow g & & \uparrow \hat{g} \\ \mathbf{A} & \xrightarrow{f} & \mathbf{B} \end{array}$$

Let us recall the well-known characterization of ages of countable homogenous structures by Fraïssé:

Theorem 2.5 (Fraïssé [15]). *Let \mathcal{C} be an age. Then \mathcal{C} is the age of a countable homogeneous structure if and only if it has the AP. Moreover, any two countable homogeneous structures with the same age are isomorphic.*

An age is called a *Fraïssé-class* if it has the AP. A countable homogeneous $\underline{\Sigma}$ -structure \mathbf{U} is called a *Fraïssé-limit* of its age $\text{Age}(\mathbf{U})$.

Example 2.6. Some examples of Fraïssé-classes include:

- the class of finite simple graphs,
- the class of finite posets (strictly or non-strictly ordered),
- the class of finite linear orders (strictly or non-strictly ordered),
- the class of finite tournaments.

The corresponding Fraïssé-limits are the Rado graph (aka the countable random graph, aka the Erdős-Rényi graph), the countable generic poset, the rationals, and the countable generic tournament, respectively.

In the following, let $\underline{\Sigma}$ be a relational signature and let $\mathcal{C}_{\underline{\Sigma}}$ be the category of all $\underline{\Sigma}$ -structures with homomorphisms as morphisms. In $\mathcal{C}_{\underline{\Sigma}}$, the amalgamated free sum is constructed as follows:

Construction. Let $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$ be $\underline{\Sigma}$ -structures, such that $\mathbf{A} \leq \mathbf{B}_1$, $\mathbf{A} \leq \mathbf{B}_2$, and such that $B_1 \cap B_2 = A$. Define $C := B_1 \cup B_2$, and for each $\varrho \in \Sigma$ define

$$\varrho^C := \varrho^{B_1} \cup \varrho^{B_2},$$

and finally $\mathbf{C} := (C, (\varrho^C)_{\varrho \in \Sigma})$. Then \mathbf{C} is called the *amalgamated free sum* of \mathbf{B}_1 with \mathbf{B}_2 with respect to \mathbf{A} . It is going to be denoted by $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$. Note that the following is always a pushout square in $\mathcal{C}_{\underline{\Sigma}}$:

$$\begin{array}{ccc} \mathbf{B}_1 & \xrightarrow{=} & \mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2 \\ \uparrow = & & \uparrow = \\ \mathbf{A} & \xrightarrow{=} & \mathbf{B}_2 \end{array}$$

Definition 2.7. We say, that the age of a $\underline{\Sigma}$ structure \mathbf{U} has the *free amalgamation property* if $\text{Age}(\mathbf{U})$ is closed with respect to amalgamated free sums in $\mathcal{C}_{\underline{\Sigma}}$.

3. AUTOMATIC HOMEOMORPHICITY

Definition 3.1. Let \mathcal{K} be a class of $\underline{\Sigma}$ -structures, and let $\mathbf{U} \in \mathcal{K}$. We say that

- $\text{Aut}(\mathbf{U})$ has *automatic homeomorphicity* with respect to \mathcal{K} if every group-isomorphism from $\text{Aut}(\mathbf{U})$ to the automorphism group of a member of \mathcal{K} is a homeomorphism,
- $\overline{\text{Aut}}(\mathbf{U})$ has *automatic homeomorphicity* with respect to \mathcal{K} if every monoid-isomorphism from $\overline{\text{Aut}}(\mathbf{U})$ to a closed submonoid of $\text{End}(\mathbf{V})$ is a homeomorphism, for every $\mathbf{V} \in \mathcal{K}$,
- $\text{End}(\mathbf{U})$ has *automatic homeomorphicity* with respect to \mathcal{K} if every monoid-isomorphism from $\text{End}(\mathbf{U})$ to the endomorphism monoid of a member of \mathcal{K} is a homeomorphism,
- $\text{Pol}(\mathbf{U})$ has *automatic homeomorphicity* with respect to \mathcal{K} if every clone-isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} is a homeomorphism.

The phrase “with respect to \mathcal{K} ” will be dropped whenever \mathcal{K} consists of all structures on U .

The notion of automatic homeomorphicity for transformation semigroups and for clones was introduced by Bodirsky, Pinsker and Pongrácz in [6]. They proved automatic homeomorphicity for the following clones:

- the Horn clone (this is the smallest closed subclone of \mathfrak{D}_ω that contains all injective functions from \mathfrak{D}_ω),
- the closed subclones of \mathfrak{D}_ω that contain $\mathfrak{D}^{(1)}$,
- the polymorphism clone of the Rado graph,
- the clone of essentially injective polymorphisms of the Rado-graph,
- the 17 minimal tractable clones over the Rado graph (cf. [4]),

To show automatic homeomorphicity for the polymorphism clone of a countable homogeneous structure \mathbf{U} with respect to a class \mathcal{K} of structures, they devised the following programme:

- (1) show that $\text{Aut}(\mathbf{U})$ has automatic homeomorphicity with respect to \mathcal{K} ,
- (2) show that $\overline{\text{Aut}}(\mathbf{U})$ has automatic homeomorphicity with respect to \mathcal{K} ,
- (3) show that every isomorphism from $\text{End}(\mathbf{U})$ to the endomorphism monoid of a member of \mathcal{K} is continuous,
- (4) show that every isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} is continuous,
- (5) show that every continuous isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} is a homeomorphism.

Step 1 of this strategy is outsourced to group theory. To be more precise, there are two standard ways to show automatic homeomorphicity for groups—the small index property (recall that a topological group is said to have the *small index property* if every subgroup of at most countable index is open, cf. [11,20,22–24,41,43]), and Rubin’s (weak) $\forall\exists$ -interpretations (cf. [2,40]). If the automorphism group of \mathbf{U} has the small index property, then $\text{Aut}(\mathbf{U})$ has automatic homeomorphicity. Moreover, if \mathbf{U} has a weak $\forall\exists$ -interpretation, then $\text{Aut}(\mathbf{U})$ has automatic homeomorphicity with respect to the class of ω -categorical structures.

Step 2 bases on [6, Lemma 12] that states that if a closed transformation monoid \mathfrak{M} on a countable set has a dense group \mathfrak{G} of units, and if only the identical endomorphism of \mathfrak{M} fixes all elements of \mathfrak{G} point-wise, then from the automatic homeomorphicity of \mathfrak{G} with respect to \mathcal{K} follows the automatic homeomorphicity of \mathfrak{M} with respect to \mathcal{K} . It is shown in [6, Theorem 21] that this criterion applies to the monoid of self-embeddings of a countable homogeneous structure \mathbf{U} whenever $\text{Aut}(\mathbf{U})$ has

automatic homeomorphicity with respect to \mathcal{K} and whenever \mathbf{U} has the *joint extension property* (cf. [6, Definition 18]).

Step 3 relies on a so called *gate technique*:

Definition 3.2 ([34, Definition 3.1], implicit in [6]). Given a transformation monoid \mathfrak{M} on a countably infinite set A . Let \mathfrak{G} be the group of units in \mathfrak{M} , and let $\overline{\mathfrak{G}}$ be the closure of \mathfrak{G} in \mathfrak{M} . Then we say that \mathfrak{M} has a *gate covering* if there exists an open covering \mathcal{U} of \mathfrak{M} and elements $f_U \in U$, for every $U \in \mathcal{U}$, such that for all $U \in \mathcal{U}$ and for all Cauchy-sequences $(g_n)_{n \in \mathbb{N}}$ of elements from U there exist Cauchy-sequences $(\kappa_n)_{n \in \mathbb{N}}$ and $(\iota_n)_{n \in \mathbb{N}}$ of elements from $\overline{\mathfrak{G}}$ such that for all $n \in \mathbb{N}$ we have

$$g_n = \kappa_n \circ f_U \circ \iota_n.$$

Now Step 3 can be fulfilled by observing that if $\overline{\text{Aut}(\mathbf{U})}$ has automatic homeomorphicity with respect to \mathcal{K} and if $\text{End}(\mathbf{U})$ has a gate covering, then every isomorphism from $\text{End}(\mathbf{U})$ to the endomorphism monoid of a member of \mathcal{K} is continuous.

Another gate-technique may be used to fulfill Step 4:

Definition 3.3 ([6, Definition 35]). Let \mathfrak{C} be a clone. Then \mathfrak{C} is said to have a *gate covering* if there exists an open covering \mathcal{U} of \mathfrak{C} and functions $f_U \in U$, for every $U \in \mathcal{U}$, such that for each $U \in \mathcal{U}$ and for all Cauchy-sequences $(g_n)_{n \in \mathbb{N}}$ of functions from U (all of the same arity k) there exist Cauchy-sequences $(\kappa_n)_{n \in \mathbb{N}}$ and $(\iota_n^i)_{n \in \mathbb{N}}$ ($i = 1, \dots, k$) of functions from $\mathfrak{C}^{(1)}$ such that

$$g_n(x_1, \dots, x_k) = \kappa_n(f_U(\iota_n^1(x_1), \dots, \iota_n^k(x_k))).$$

In [6, Theorem 37] it is shown that whenever $\text{Pol}(\mathbf{U})$ has a gate covering then every isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} , whose restriction to $\text{End}(\mathbf{U})$ is continuous, is itself continuous.

Finally, in Step 5 a topological version of Birkhoff's theorem from [5] is used to show that every continuous isomorphism from $\text{Pol}(\mathbf{U})$ to the Polymorphism clone of some structure from \mathcal{K} is open, too.

The above sketched strategy was used in [6] for showing automatic homeomorphicity of the polymorphism clone of the Rado graph.

Each of the 5 steps carries substantial difficulties. In the following we are going to short-circuit this process, by proving automatic homeomorphicity of the polymorphism clone of a structure \mathbf{U} without showing first the automatic homeomorphicity of $\overline{\text{Aut}(\mathbf{U})}$ and/or $\text{End}(\mathbf{U})$.

In particular, we devise two new strategies for showing automatic homeomorphicity for the polymorphism clone of a countable homogeneous structure \mathbf{U} with respect to a class \mathcal{K} of structures:

First strategy

- (1) Show that every continuous isomorphism from the polymorphism clone of a member of \mathcal{K} to $\text{Pol}(\mathbf{U})$ is a homeomorphism.
- (2) Show that every isomorphism from the polymorphism clone of a member of \mathcal{K} to $\text{Pol}(\mathbf{U})$ is continuous.

Second strategy

- (1) Show that $\text{Aut}(\mathbf{U})$ has automatic homeomorphicity with respect to \mathcal{K} .
- (2) Show that $\overline{\text{Aut}(\mathbf{U})}$ has automatic homeomorphicity with respect to \mathcal{K} .
- (3) Show that every isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} is continuous.
- (4) Show that every continuous isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of another member of \mathcal{K} is a homeomorphism.

Both our strategies base on a gate-technique: The following definition is a slightly stronger formulation of Definition 3.3 in the spirit of Definition 3.2:

Definition 3.4. Let \mathfrak{C} be a clone, let \mathfrak{G} be the group of units in $\mathfrak{C}^{(1)}$, and let $\overline{\mathfrak{G}}$ be the closure of \mathfrak{G} in $\mathfrak{C}^{(1)}$. Then \mathfrak{C} is said to have a *strong gate covering* if there exists an open covering \mathcal{U} of \mathfrak{C} and functions $f_U \in U$, for every $U \in \mathcal{U}$, such that for each $U \in \mathcal{U}$ and for all Cauchy-sequences $(g_n)_{n \in \mathbb{N}}$ of functions from U (each of the same arity k) there exist Cauchy-sequences $(\kappa_n)_{n \in \mathbb{N}}$ and $(\iota_n^i)_{n \in \mathbb{N}}$ ($i = 1, \dots, k$) of functions from $\overline{\mathfrak{G}}$ such that

$$g_n(x_1, \dots, x_k) = \kappa_n(f_U(\iota_n^1(x_1), \dots, \iota_n^k(x_k))).$$

Strong gate coverings allow to lift continuity properties:

Lemma 3.5. Let \mathbf{A} and \mathbf{B} be two countable relational structures, such that $\text{Pol}(\mathbf{A})$ has a strong gate covering. Let $h: \text{Pol}(\mathbf{A}) \rightarrow \text{Pol}(\mathbf{B})$ be a clone homomorphism whose restriction to $\overline{\text{Aut}(\mathbf{A})}$ is continuous. Then h is continuous, too.

Proof. Let $(v_n)_{n \in \mathbb{N}}$ be a Cauchy-sequence of k -ary polymorphisms of \mathbf{A} . Since $(\text{Pol}(\mathbf{A}), d_A)$ is complete, $(v_n)_{n \in \mathbb{N}}$ is convergent—say to $v \in \text{Pol}^{(k)}(\mathbf{A})$.

Let $(\mathcal{U}, (f_U)_{U \in \mathcal{U}})$ be a strong gate covering of $\text{Pol}(\mathbf{A})$. Then there exists a $U \in \mathcal{U}$ and an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $v_n \in U$. Without loss of generality, assume that $n_0 = 0$. By the definition of strong gate coverings there exist Cauchy-sequences $(\kappa_n)_{n \in \mathbb{N}}$ and $(\iota_n^i)_{n \in \mathbb{N}}$ ($i = 1, \dots, k$) in $\overline{\text{Aut}(\mathbf{A})}$, such that

$$v_n(x_1, \dots, x_k) = \kappa_n(f_U(\iota_n^1(x_1), \dots, \iota_n^k(x_k))),$$

for all $n \in \mathbb{N}$. In particular, with $\kappa = \lim_{n \rightarrow \infty} \kappa_n$ and $\iota^i = \lim_{n \rightarrow \infty} \iota_n^i$, we have

$$v(x_1, \dots, x_k) = \kappa(f_U(\iota^1(x_1), \dots, \iota^k(x_k))).$$

Because $h|_{\overline{\text{Aut}(\mathbf{A})}}$ is continuous, we have $\lim_{n \rightarrow \infty} h(\kappa_n) = h(\kappa)$ and $\lim_{n \rightarrow \infty} h(\iota_n^i) = h(\iota^i)$, for all $i = 1 \dots k$.

Now, since h is a clone-isomorphism, we have

$$h(v_n)(x_1, \dots, x_k) = h(\kappa_n)(h(f_U)(h(\iota_n^1)(x_1), \dots, h(\iota_n^k)(x_k))).$$

Thus, since the composition of functions is continuous, we have that the sequence $(h(v_n))_{n \in \mathbb{N}}$ converges to $h(v)$. From this, it follows that h is continuous. \square

3.1. About the first strategy.

Proposition 3.6. Let \mathbf{A} and \mathbf{B} be two countable relational structures, such that $\text{Pol}(\mathbf{B})$ has a strong gate covering. Let $h: \text{Pol}(\mathbf{A}) \rightarrow \text{Pol}(\mathbf{B})$ be a continuous clone-isomorphism. Then h is a homeomorphism.

Before coming to the proof of this proposition, let us make some auxiliary observations:

Lemma 3.7. Let A, B be countable sets, and let $\mathfrak{M}_1 \leq \mathfrak{D}_A^{(1)}$, $\mathfrak{M}_2 \leq \mathfrak{D}_B^{(1)}$ be monoids, such that \mathfrak{M}_1 has a dense set of units. Let $h: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ be a continuous homomorphism. Then h is uniformly continuous from (\mathfrak{M}_1, d_A) to (\mathfrak{M}_2, d_B) .

Proof. Suppose that the metrics d_A and d_B are induced by enumerations \bar{a} and \bar{b} of A and B , respectively. Let e_1, e_2 be the neutral elements of \mathfrak{M}_1 and of \mathfrak{M}_2 , respectively. Let $\varepsilon > 0$. Since h is continuous at e_1 , there exists a $\Delta \in \mathbb{N} \setminus \{0\}$ such that, with $\delta := 2^{-\Delta}$, for all $m \in \mathfrak{M}_1$ with $d_{\bar{a}}(m, e_1) \leq \delta$ we have $d_{\bar{b}}(h(m), e_2) \leq \varepsilon$.

Let $m, m' \in \mathfrak{M}_1$ with $d_A(m, m') \leq \delta$. Then we have

$$(m(a_0), \dots, m(a_{\Delta-1})) = (m'(a_0), \dots, m'(a_{\Delta-1})) =: \bar{c}.$$

But since the units lie dense in \mathfrak{M}_1 , there exists a unit $g \in \mathfrak{M}_1$ with

$$(g(a_0), \dots, g(a_{\Delta-1})) = \bar{c}.$$

Consider now $\tilde{m} := g^{-1} \circ m$ and $\tilde{m}' := g^{-1} \circ m'$. Then $d_A(\tilde{m}, e_1) \leq \delta$ and $d_A(\tilde{m}', e_1) \leq \delta$.

Now we compute

$$\begin{aligned} \varepsilon &\geq d_B(h(\tilde{m}), e_2) = d_B(h(g^{-1} \circ m), e_2) = d_B(h(g)^{-1} \circ h(m), e_2) \\ &= d_B(h(m), h(g)) \end{aligned}$$

In the same way we obtain $d_B(h(m'), h(g)) \leq \varepsilon$. Since d_B is an ultrametric, we finally conclude that $d_B(h(m), h(m')) \leq \varepsilon$. \square

We will further need the following basic facts about metric spaces and uniform continuous functions:

Lemma 3.8 (Hausdorff [19, Page 368]). *Let (M_1, d_1) be a metric space and let (M_2, d_2) be a complete metric space. Then every uniformly continuous function $f: (M_1, d_1) \rightarrow (M_2, d_2)$ has a unique uniformly continuous extension to the completion of (M_1, d_1) .*

Proof. This is folklore. \square

Corollary 3.9. *Let Met be the category of metric spaces with uniformly continuous functions. Let cMet be the full subcategory of Met spanned by all complete metric spaces. Then the assignment that maps every metric space \mathbb{M} to its completion $\widehat{\mathbb{M}}$ and that maps every uniform continuous function $f: \mathbb{M}_1 \rightarrow \mathbb{M}_2$ to its unique extension $\hat{f}: \widehat{\mathbb{M}}_1 \rightarrow \widehat{\mathbb{M}}_2$ is a functor from Met to cMet .*

Proof. This is folklore. \square

Remark. In fact, cMet is a reflective subcategory of Met , and the completion functor is the corresponding reflector. This is one of the earliest examples reflective subcategories. In Freyd's PhD-thesis (this is the place where Freyd introduced notion of reflective subcategories) it is shown that the class of complete metric spaces induces a reflective subcategory in the category of metric spaces with non-expansive mappings (cf. [17, Page 25]). The same proof functions for the situation with uniformly continuous functions (cf. [16, Page 79]).

We are going to denote the completion functor by C . Finally we are going to make use of the following observation by Lascar:

Proposition 3.10 ([28, Corollary 2.8]). *Let \mathbf{A} and \mathbf{B} be countable relational structures and let f be a continuous isomorphism from $\text{Aut}(\mathbf{A})$ to $\text{Aut}(\mathbf{B})$. Then f is a homeomorphism.*

Eventually we can come to the proof of Proposition 3.6:

Proof of Proposition 3.6. Let $f := h|_{\text{Aut}(\mathbf{A})}$. Since h is continuous, we have that f is continuous, too. Thus, by Proposition 3.10, f is a homeomorphism. By Lemma 3.7, $f: (\text{Aut}(\mathbf{A}), d_A) \rightarrow (\text{Aut}(\mathbf{B}), d_B)$ and $f^{-1}: (\text{Aut}(\mathbf{B}), d_B) \rightarrow (\text{Aut}(\mathbf{A}), d_A)$ are uniformly continuous. That is, f is an isomorphism in the category Met . Let $\hat{f} := C(f)$ be the unique uniformly continuous extension of f to $\overline{\text{Aut}(\mathbf{A})}$. Then, since C is a functor, we have that $\hat{f}: \overline{\text{Aut}(\mathbf{A})} \rightarrow \overline{\text{Aut}(\mathbf{B})}$ is an isomorphism in the category cMet , and in particular we have that $C(f^{-1}) = C(f)^{-1} = \hat{f}^{-1}$ holds.

Let now $g := h|_{\overline{\text{Aut}(\mathbf{A})}}$. Since h is continuous, it follows that $g: \overline{\text{Aut}(\mathbf{A})} \rightarrow \overline{\text{Aut}(\mathbf{B})}$ is continuous, too. Thus, from Lemma 3.7 we conclude that $g: (\overline{\text{Aut}(\mathbf{A})}, d_A) \rightarrow (\overline{\text{Aut}(\mathbf{B})}, d_B)$ is uniformly

continuous. Because, clearly, we have $g|_{\text{Aut}(\mathbf{A})} = f$, we conclude from Lemma 3.8, that $g = C(f) = \hat{f}$. Thus $g: \overline{\text{Aut}(\mathbf{A})} \rightarrow \overline{\text{Aut}(\mathbf{B})}$ is a homeomorphism.

Now, since h^{-1} is a clone-homomorphism, and since $(h^{-1})|_{\overline{\text{Aut}(\mathbf{B})}} = g^{-1}$, and since g^{-1} is continuous, it follows from Lemma 3.5 that h^{-1} is continuous, too. \square

Corollary 3.11. *Let \mathcal{K} be a class of structures and let $\mathbf{U} \in \mathcal{K}$, such that $\text{Pol}(\mathbf{U})$ has a strong gate covering. Then $\text{Pol}(\mathbf{U})$ has automatic homeomorphicity with respect to \mathcal{K} if and only if every isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} is open.*

Proof. Suppose that every isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} is open. Let $\mathbf{V} \in \mathcal{K}$, and let $h: \text{Pol}(\mathbf{U}) \rightarrow \text{Pol}(\mathbf{V})$ be an isomorphism. Then h is open. Hence $h^{-1}: \text{Pol}(\mathbf{V}) \rightarrow \text{Pol}(\mathbf{U})$ is a continuous clone isomorphism. Since $\text{Pol}(\mathbf{U})$ has a strong gate covering, it follows from Proposition 3.6, that h^{-1} is a homeomorphism. Thus, h is a homeomorphism, too.

The proof of the other direction of the claim is trivial. \square

In order to fulfill our first strategy, we may use the following results from [6]:

Proposition 3.12 ([6, Proposition 27]). *Let \mathbf{U} be a relational structure such that $\text{Pol}(\mathbf{U})$ contains all constant functions. Then every isomorphism from $\text{Pol}(\mathbf{U})$ to another clone of functions is open.*

If it is known that $\text{End}(\mathbf{U})$ has automatic homeomorphicity with respect to \mathcal{K} , then there is an alternative to show openness for the isomorphisms from $\text{Pol}(\mathbf{U})$ to polymorphism clones of structures from \mathcal{K} , in case that $\text{Aut}(\mathbf{U})$ acts transitively on U :

Proposition 3.13 ([6, Proposition 32]). *If $\text{Aut}(\mathbf{U})$ is transitive, then every injective clone homomorphism h from $\text{Pol}(\mathbf{U})$ to another clone, whose restriction to $\text{End}(\mathbf{U})$ is open, is itself open.*

Remark. Note that our first strategy does not require us to show automatic homeomorphicity of $\text{Aut}(\mathbf{U})$, $\overline{\text{Aut}(\mathbf{U})}$, or $\text{End}(\mathbf{U})$, in order to derive the automatic homeomorphicity of $\text{Pol}(\mathbf{U})$.

3.2. About the second strategy. Our second strategy uses, apart from strong gate coverings, a technique from [6], that was used there in order to show automatic homeomorphicity of the polymorphism clone of the Rado graph. We are going to make this technique applicable to a much wider class of relational structures. The key is going to be a topological version of Birkhoff's theorem due to Bodirsky and Pinsker:

Theorem 3.14 ([5, Theorem 4]). *Let \mathbb{A} and \mathbb{B} be countable algebras over the same signature, whose clones of term functions are \mathfrak{A} and \mathfrak{B} , respectively. Suppose that $\overline{\mathfrak{A}}^{(1)}$ has an oligomorphic group of units and that \mathbb{B} is finitely generated. Then the following are equivalent:*

- (1) $\mathbb{B} \in \text{HSP}^{\text{fin}}(\mathbb{A})$,
- (2) the clone homomorphism $\xi: \mathfrak{A} \rightarrow \mathfrak{B}$ that maps $f^{\mathbb{A}}$ to $f^{\mathbb{B}}$, for all basic operations f , exists and is continuous.

Before being able to state the main result of this subsection, another, by now well-established property of ages of relational structures needs to enter the stage—the homo-amalgamation property (HAP):

Definition 3.15. Let \mathcal{C} be a class of Σ -structures. We say that \mathcal{C} has the *homo-amalgamation property* (HAP) if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ from \mathcal{C} , for all homomorphisms $f: \mathbf{A} \rightarrow \mathbf{B}$, and for all embeddings $g: \mathbf{A} \hookrightarrow \mathbf{C}$, there exists $\mathbf{D} \in \mathcal{C}$, a homomorphism $\hat{f}: \mathbf{C} \hookrightarrow \mathbf{D}$, and an embeddings $\hat{g}: \mathbf{B} \hookrightarrow \mathbf{D}$ such

that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\hat{f}} & \mathbf{D} \\ \uparrow g & & \uparrow \hat{g} \\ \mathbf{A} & \xrightarrow{f} & \mathbf{B} \end{array}$$

In the rest of this subsection, we are going to prove the following result:

Proposition 3.16. *Let \mathbf{U} be a countable, homogeneous, ω -categorical relational structure such that*

- (1) $\text{Aut}(\mathbf{U})$ acts transitively on U ,
- (2) $\text{Age}(\mathbf{U})$ has the free amalgamation property,
- (3) $\text{Age}(\mathbf{U})$ is closed with respect to finite products,
- (4) $\text{Age}(\mathbf{U})$ has the HAP.

Then every continuous isomorphism from $\text{Pol}(\mathbf{U})$ to another closed subclone \mathcal{D} of $\mathfrak{D}_{\mathbf{U}}$ is a homeomorphism.

As usual, before proving this proposition, let us collect the necessary tools: Recall that a consistent set of primitive positive formulae with free variables in $\{x_1, \dots, x_n\}$ is called *aprimitive positive n -type*. To a structure \mathbf{A} and a relation $\sigma \subseteq A^n$ we may associate a primitive positive type according to

$$\text{Tp}_{\mathbf{A}}(\sigma) := \{\varphi(x_1, \dots, x_n) \mid \forall \bar{a} \in \sigma : \mathbf{A} \models \varphi(\bar{a})\}.$$

primitive positive types that arise in this way are called *closed*. A primitive positive n -type Ψ is called *complete* if there exists a structure \mathbf{A} and a finite relation $\sigma \subseteq A^n$, such that $\Psi = \text{Tp}_{\mathbf{A}}(\sigma)$.

Recall also that a structure is called *weakly oligomorphic* if its endomorphism monoid has just finitely many invariant relations of every given finite arity [31]. By a result by Mašulovic [30], a countable structure \mathbf{A} is weakly oligomorphic if and only if its polymorphism clone has just finitely many invariant relations of every finite arity (cf. also [35, Proposition 4.8]). Finally, by [35, Proposition 4.7], \mathbf{A} is weakly oligomorphic, if and only if it affords just finitely many closed primitive positive types of every finite arity. Note that this implies immediately that in a countable weakly oligomorphic structure all closed primitive positive types are complete.

Lemma 3.17. *Let \mathbf{A} be a weakly oligomorphic relational structure with quantifier elimination for primitive positive formulae, whose age is closed with respect to finite products. Then every complete primitive positive type Φ over \mathbf{A} is of the shape $\text{Tp}_{\mathbf{A}}(\bar{a})$ for a suitable tuple \bar{a} of elements of A .*

Proof. Let Φ be an m -ary complete primitive positive type over \mathbf{A} . Then, since \mathbf{A} is weakly oligomorphic, there exists $\{\bar{a}_1, \dots, \bar{a}_n\} \subseteq A^m$ such that $\Phi = \text{Tp}_{\mathbf{A}}(\{\bar{a}_1, \dots, \bar{a}_n\})$. Suppose $\bar{a}_j = (a_{1,j}, \dots, a_{m,j})$ for $j \in \{1, \dots, n\}$. Let $\bar{b}_i := (a_{i,1}, \dots, a_{i,n})$, for $i \in \{1, \dots, m\}$. Let \mathbf{B} be the substructure of \mathbf{A}^n spanned by $\{\bar{b}_1, \dots, \bar{b}_m\}$. Since $\text{Age}(\mathbf{A})$ is closed with respect to finite products, we have $\mathbf{B} \in \text{Age}(\mathbf{A})$. Let $\iota : \mathbf{B} \hookrightarrow \mathbf{A}$ be an embedding from \mathbf{B} into \mathbf{A} , and let $c_i := \iota(\bar{b}_i)$, for $i \in \{1, \dots, m\}$. Then $\text{Tp}_{\mathbf{A}}((c_1, \dots, c_n))$ contains the same atomic formulae like Φ . Since \mathbf{A} has quantifier elimination for primitive positive formulae, we have $\Phi = \text{Tp}_{\mathbf{A}}((c_1, \dots, c_n))$. \square

Proposition 3.18. *Let \mathbf{U} be a countable, homogeneous, ω -categorical relational structure with quantifier elimination for primitive positive formulae such that*

- (1) $\text{Aut}(\mathbf{U})$ acts transitively on U ,
- (2) $\text{Age}(\mathbf{U})$ has the free amalgamation property,
- (3) $\text{Age}(\mathbf{U})$ is closed with respect to finite products.

Then every continuous isomorphism to another closed subclone \mathfrak{D} of \mathfrak{D}_U is a homeomorphism.

Proof. The proof follows the lines of the proof of [6, Lemma 49], where our claim is proved for the special case when \mathbf{U} is the Rado graph. Let $\xi: \text{Pol}(\mathbf{U}) \rightarrow \mathfrak{D}$ be a continuous clone-isomorphism.

First, for every $n \in \mathbb{N} \setminus \{0\}$, and for every $f \in \text{Pol}^{(n)}(\mathbf{U})$, let \underline{f} be an n -ary operation symbol. Let $\underline{\Sigma}$ be the algebraic signature, that consists of all newly defined operation symbols. Now we consider the algebras $\mathbb{U} = (U, \text{Pol}(\mathbf{U}))$, $\mathbb{D} = (U, \mathfrak{D})$ as $\underline{\Sigma}$ -algebras, where for every $\underline{f} \in \underline{\Sigma}$ the interpretation of \underline{f} in \mathbb{U} is f and the interpretation of \underline{f} in \mathbb{D} is $\xi(f)$.

Let \mathbb{B} be some finitely generated subalgebra of \mathbb{D} with at least two elements, and let $r: \mathfrak{D} \rightarrow \mathfrak{D}_B$ be the restriction homomorphism defined by $r(g) := g|_B$. Let \mathfrak{D}_B be the image of \mathfrak{D} under r . Then $\mathbb{B} = (B, \mathfrak{D}_B)$, where $\underline{f} \in \underline{\Sigma}$ is interpreted as $r(\xi(f))$, for all $f \in \underline{\Sigma}$.

Since (B, \mathfrak{D}_B) is a subalgebra of (U, \mathfrak{D}) , it follows from the topological Birkhoff theorem that $r: \mathfrak{D} \rightarrow \mathfrak{D}_B$ is a continuous clone-homomorphism.

In the following, we will show that $\xi' := r \circ \xi$ is a homeomorphism. When this is done, it follows that ξ is a homeomorphism, too, since in this case we have that r is bijective, thus r^{-1} is an open clone isomorphism, and thus $\xi = r^{-1} \circ \xi'$ is open.

Since ξ' is a continuous clone-homomorphism, and since \mathbb{B} is finitely generated, it follows from the topological Birkhoff theorem that \mathbb{B} is contained in the pseudovariety generated by \mathbb{U} . In other words, \mathbb{B} is a homomorphic image of a subalgebra of a finite power of \mathbb{U} . Let \mathbb{S} be the corresponding subalgebra in this process, and let \sim be the kernel of the surjective homomorphism from \mathbb{S} to \mathbb{B} . Then for some n , we have that S is an n -ary invariant relation of $\text{Pol}(\mathbf{U})$. Since \mathbf{U} is ω -categorical, it follows from [3, Theorem 4], that S is definable by a set Ψ of primitive positive formulae in the language of \mathbf{U} . We may suppose without loss of generality that $\Psi = \text{Tpp}_{\mathbf{U}}(S)$. Also, without loss of generality, we may assume that Ψ does not contain a formula of the shape $x_i = x_j$ for $i \neq j$. Thus, by Lemma 3.17, S contains at least one irreflexive tuple.

The relation \sim is a congruence relation of the algebra \mathbb{S} , i.e., it is invariant under all term-functions of \mathbb{S} . Note that the term functions of \mathbb{S} are just the elements of $\text{Pol}(\mathbf{U})$ in their natural action on n -tuples. Thus, if we consider

$$\sigma^\sim := \{\bar{u}\bar{v} \mid \bar{u}, \bar{v} \in S, \bar{u} \sim \bar{v}\},$$

then σ^\sim is a $2n$ -ary invariant relation of $\text{Pol}(\mathbf{U})$. By the same reasoning as above, σ^\sim is defined through a set Φ of primitive positive formulae over \mathbf{U} . Again, we may assume that $\Phi = \text{Tpp}_{\mathbf{U}}(\sigma^\sim)$. To improve readability, we use the following convention for the names of the variables in formulae from Φ : Every formula in $\varphi \in \Phi$ shall be of the form $\varphi(\bar{x}, \bar{y})$, where $\bar{x} = (x_1, \dots, x_n)$ and where $\bar{y} = (y_1, \dots, y_n)$. Clearly, because \sim is reflexive and symmetric, if $\varphi(\bar{x}, \bar{y}) \in \Phi$, then we also have $\varphi(\bar{x}, \bar{x}) \in \Psi$ and $\varphi(\bar{y}, \bar{x}) \in \Phi$.

Observe that Φ does not contain a formula of the shape $x_i = y_j$, for $i \neq j$, for otherwise we would obtain $x_i = x_j \in \Psi$ —contradictory with our assumptions on Ψ .

We are now going to show that Φ necessarily contains a formula $x_i = y_i$, for some $i \in \{1, \dots, n\}$. Suppose that Φ does not contain any such formula. Since \sim has more than one equivalence class, and since \mathbf{U} has quantifier elimination for primitive positive formulae, Φ contains an atomic formula $\varphi(\bar{x}, \bar{y}) = \varrho(z_1, \dots, z_k)$, where $z_1, \dots, z_k \in \{x_1, \dots, x_n, y_1, \dots, y_n\}$, and where $\{z_1, \dots, z_k\} \cap \{x_1, \dots, x_n\}$ and $\{z_1, \dots, z_k\} \cap \{y_1, \dots, y_n\}$ are both nonempty. By Lemma 3.17, there exists $\bar{u}\bar{v} \in \sigma^\sim$, such that $\text{Tpp}_{\mathbf{U}}(\bar{u}\bar{v}) = \Phi$. Moreover, we have $\text{Tpp}_{\mathbf{U}}(\bar{u}) = \text{Tpp}_{\mathbf{U}}(\bar{v}) = \Psi$. Let \mathbf{U} and \mathbf{W} the substructures of \mathbf{U} induced by $U = \{u_1, \dots, u_n\}$ and $W = U \cup \{v_1, \dots, v_n\}$, respectively. Let \mathbf{W}' be an isomorphic copy of \mathbf{W} such that $W' = U \cup \{v'_1, \dots, v'_n\}$ and such that $W \cap W' = U$ and are disjoint and such that $\iota: \mathbf{W} \rightarrow \mathbf{W}'$ defined through $\iota': u_i \mapsto u_i, v_i \mapsto v'_i$ is an isomorphism. Then, since $\text{Age}(\mathbf{U})$ has the free amalgamation property, we have that $\mathbf{W} \oplus_{\mathbf{U}} \mathbf{W}' \in \text{Age}(\mathbf{U})$. Thus,

we can assume that $\mathbf{W} \oplus_{\mathbf{U}} \mathbf{W}' \leq \mathbf{U}$. Let $\bar{v}' := (v'_1, \dots, v'_n)$. Then by construction we have that $\text{Tp}_{\mathbf{U}}^{(0)}(\bar{u}\bar{v}) = \text{Tp}_{\mathbf{U}}^{(0)}(\bar{u}\bar{v}')$. Since \mathbf{U} has quantifier elimination for primitive positive formulae, we also have $\text{Tp}_{\mathbf{U}}(\bar{u}\bar{v}) = \text{Tp}_{\mathbf{U}}(\bar{u}\bar{v}')$. Hence, $\bar{u} \sim \bar{v}'$. Since \sim is symmetric and transitive, we have $\bar{v} \sim \bar{v}'$. Thus, we have $\varphi(\bar{x}, \bar{y}) \in \text{Tp}_{\mathbf{U}}(\bar{v}\bar{v}')$. However, by the nature of the amalgamated free sum in free amalgamation classes, we have that $\varrho^{\mathbf{W}} \cap \{v_1, \dots, v_n, v'_1, \dots, v'_n\}^k = \emptyset$. With $\varrho^{\mathbf{W}} = \varrho^{\mathbf{U}} \cap W^k$, we arrive at a contradiction. Thus, our assumption was wrong and Φ contains a formula $x_{i_0} = y_{i_0}$ for some $i_0 \in \{1, \dots, n\}$.

Next we show that ξ' is injective. Without loss of generality we may assume that \mathbb{B} is equal to \mathbb{S}/\sim . Let $f, g \in \text{Pol}^{(m)}(\mathbf{U})$ be two distinct functions. Then there exists $\bar{a} = (a_1, \dots, a_m) \in U^m$, such that

$$b := f(a_1, \dots, a_m) \neq g(a_1, \dots, a_m) =: b'$$

Since $\text{Aut}(\mathbf{U})$ acts transitively on U , there exist

$$\bar{c}_1 = (c_{1,1}, \dots, c_{n,1}), \dots, \bar{c}_m = (c_{1,m}, \dots, c_{n,m}) \in S,$$

such that $c_{i_0,j} = a_j$, for each $j \in \{1, \dots, m\}$. Let

$$\begin{aligned} \bar{b} &= \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} f(c_{1,1}, \dots, c_{1,m}) \\ \vdots \\ f(c_{n,1}, \dots, c_{n,m}) \end{pmatrix}, \text{ and} \\ \bar{b}' &= \begin{pmatrix} b'_1 \\ \vdots \\ b'_n \end{pmatrix} := \begin{pmatrix} g(c_{1,1}, \dots, c_{1,m}) \\ \vdots \\ g(c_{n,1}, \dots, c_{n,m}) \end{pmatrix}. \end{aligned}$$

Then $b_{i_0} = b \neq b' = b'_{i_0}$. Hence

$$\xi'(f)([c_1]_{\sim}, \dots, [c_n]_{\sim}) = [\bar{b}]_{\sim} \neq [\bar{b}']_{\sim} = \xi'(g)([c_1]_{\sim}, \dots, [c_n]_{\sim}).$$

Thus, ξ' is injective (and hence bijective).

It remains to show that ξ' is open. Let $a_0, \dots, a_k \in U$, and let N be the basic clopen subset of $\text{Pol}(\mathbf{U})$ that consists of all functions $f \in \text{Pol}^{(k)}(\mathbf{U})$ with the property that $f(a_1, \dots, a_k) = a_0$. Let us define

$$A := \{[(b_1, \dots, b_n)]_{\sim} \in S/\sim \mid b_{i_0} = a_0\}.$$

Because $\text{Aut}(\mathbf{U})$ acts transitively on U , it follows that A is non-empty. For every $j \in \{1, \dots, k\}$ let $\bar{c}_j = (c_{j,1}, \dots, c_{j,n})$ be an element of S , such that $c_{j,i_0} = a_j$. Again, the existence of these tuples follows from the transitivity of $\text{Aut}(\mathbf{U})$. We are going to show now that for all $f \in \text{Pol}^{(k)}(\mathbf{U})$ we have

$$f(a_1, \dots, a_k) = a_0 \iff \xi'(f)([\bar{c}_1]_{\sim}, \dots, [\bar{c}_k]_{\sim}) \in A.$$

Indeed, if $f(a_1, \dots, a_k) = a_0$, then

$$\xi'(f)([\bar{c}_1]_{\sim}, \dots, [\bar{c}_k]_{\sim}) = \begin{bmatrix} f(c_{1,1}, \dots, c_{1,k}) \\ \vdots \\ f(c_{i_0,1}, \dots, c_{i_0,k}) \\ \vdots \\ f(c_{n,1}, \dots, c_{n,k}) \end{bmatrix}_{\sim} = \begin{bmatrix} f(c_{1,1}, \dots, c_{1,k}) \\ \vdots \\ f(a_1, \dots, a_k) \\ \vdots \\ f(c_{n,1}, \dots, c_{n,k}) \end{bmatrix}_{\sim}$$

Thus, $\xi'(f)([\bar{c}_1]_{\sim}, \dots, [\bar{c}_k]_{\sim}) \in A$.

If on the other hand $\xi'(f)([\bar{c}_1]_{\sim}, \dots, [\bar{c}_k]_{\sim}) \in A$, then

$$f(a_1, \dots, a_k) = f(c_{i_0,1}, \dots, c_{i_0,k}) = a_0.$$

Thus, we obtain that

$$\xi'(N) = \bigcup_{[\bar{c}_0]_{\sim} \in A} \{\xi'(f) \mid f \in \text{Pol}^{(k)}(\mathbf{U}), \xi'(f)([c_1]_{\sim}, \dots, [c_k]_{\sim}) = [c_0]_{\sim}\}.$$

Hence $\xi'(N)$ is open. This finishes the proof that ξ' is open. \square

In order to make Proposition 3.18 applicable, we need a convenient for a relational structure to have quantifier elimination for primitive positive formulae:

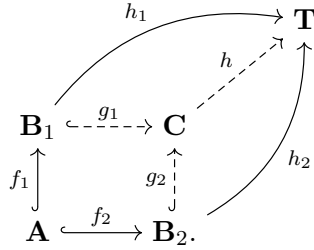
Proposition 3.19. *Let \mathbf{U} be a countable homogeneous ω -categorical relational structure such that*

- (1) $\text{Age}(\mathbf{U})$ has the free amalgamation property,
- (2) $\text{Age}(\mathbf{U})$ is closed with respect to finite products,
- (3) $\text{Age}(\mathbf{U})$ has the HAP.

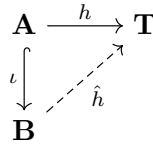
Then \mathbf{U} has quantifier elimination for primitive positive formulae.

Proposition 3.20. *Let \mathbf{U} be a countable homogeneous relational structure, and let $\mathbf{T} \in \overline{\text{Age}(\mathbf{U})}$, such that*

- (1) *for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \text{Age}(\mathbf{U})$, $f_1: \mathbf{A} \hookrightarrow \mathbf{B}_1$, $f_2: \mathbf{A} \hookrightarrow \mathbf{B}_2$, $h_1: \mathbf{B}_1 \rightarrow \mathbf{T}$, $h_2: \mathbf{B}_2 \rightarrow \mathbf{T}$, if $h_1 \circ f_1 = h_2 \circ f_2$, then there exists $\mathbf{C} \in \text{Age}(\mathbf{U})$, $g_1: \mathbf{B}_1 \hookrightarrow \mathbf{C}$, $g_2: \mathbf{B}_2 \hookrightarrow \mathbf{C}$, $h: \mathbf{C} \rightarrow \mathbf{T}$, such that the following diagram commutes:*



- (2) *for all $\mathbf{A}, \mathbf{B} \in \text{Age}(\mathbf{U})$, $\iota: \mathbf{A} \hookrightarrow \mathbf{B}$, $h: \mathbf{A} \rightarrow \mathbf{T}$ there exists $\hat{h}: \mathbf{B} \rightarrow \mathbf{T}$ such that the following diagram commutes:*



Then \mathbf{T} is isomorphic to a retract of \mathbf{U} .

Proof. This follows directly from [33, Theorem 4.2]. \square

Proof of Proposition 3.19. We are going to show that \mathbf{U} is polymorphism homogeneous (in the sense of [35]). Then it follows from [35, Corollary 3.13] and the assumption that \mathbf{U} is ω -categorical, that \mathbf{U} has quantifier elimination for primitive positive formulae.

In order to show that \mathbf{U} is polymorphism homogeneous, we are going to show that all finite powers of \mathbf{U} are homomorphism homogeneous. After that it follows from [35, Proposition 2.1], that \mathbf{U} is polymorphism homogeneous.

In order to show that every finite power of \mathbf{U} is homomorphism homogeneous, we are first going to argue that \mathbf{U} is homomorphism homogeneous (this follows from [13, Proposition 3.8]; note that the IPHEP mentioned in this paper is equivalent to the HAP). Then we will show that every finite power

of \mathbf{U} is in fact isomorphic to a retract of \mathbf{U} . Finally, it follows from the folklore fact that retracts of homomorphism-homogeneous structures are homomorphism homogeneous, that all finite powers of \mathbf{U} are homomorphism homogeneous.

In order to show that every finite power of \mathbf{U} is isomorphic to a retract of \mathbf{U} , we will make use of Proposition 3.20. First of all, since $\text{Age}(\mathbf{U})$ has the free amalgamation property, condition 1 of Proposition 3.20 is satisfied for every structure \mathbf{T} , younger than \mathbf{U} . We simply need to choose \mathbf{C} to be equal to $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$.

Let us verify condition 2 of Proposition 3.20, when $\mathbf{T} = \mathbf{U}^n$: Let $\mathbf{A}, \mathbf{B} \in \text{Age}(\mathbf{U})$, let $\iota: \mathbf{A} \hookrightarrow \mathbf{B}$ be an embedding, and let $h: \mathbf{A} \rightarrow \mathbf{U}^n$ be a homomorphism. For every $i \in \{1, \dots, n\}$ let $h_i: \mathbf{A} \rightarrow \mathbf{U}$ be defined through $h_i := e_i^n \circ h$. Since \mathbf{U} is homomorphism homogeneous, it follows that it is also weakly homomorphism homogeneous. Thus, for every $i \in \{1, \dots, n\}$, there exists a homomorphism $\hat{h}_i: \mathbf{B} \rightarrow \mathbf{U}$, such that $\hat{h}_i \circ \iota = h_i$. Now we may define \hat{h} according to

$$\hat{h} := \langle h_1, \dots, h_n \rangle: \mathbf{B} \rightarrow \mathbf{U}^n: b \mapsto (\hat{h}_1(b), \dots, \hat{h}_n(b)).$$

Clearly, with this definition we have $\hat{h} \circ \iota = h$. Thus, we may apply Proposition 3.20 to the case $\mathbf{T} = \mathbf{U}^n$, and we obtain that \mathbf{U}^n is isomorphic to a retract of \mathbf{U} . \square

Proof of Proposition 3.16. This immediately follows from Proposition 3.18, together with Proposition 3.19. \square

Remark. Retracts of homogeneous structures were considered also by Dolinka and Kubiś ([12,26]).

3.3. Existence of strong gate coverings. The hardest part in both our strategies for showing automatic homeomorphicity is to prove the existence of a strong gate covering. A major part of the rest of the paper will be devoted to this task.

Definition 3.21. Let \mathbf{U} be a structure. An n -ary polymorphism u of \mathbf{U} is called *universal* if for all structures $\mathbf{A} \in \overline{\text{Age}}(\mathbf{U})$ and for every homomorphism $f: \mathbf{A}^n \rightarrow \mathbf{U}$ there exist $\iota: \mathbf{A} \hookrightarrow \mathbf{U}$ such that for all $(a_1, \dots, a_n) \in \mathbf{A}^n$ holds $f(a_1, \dots, a_n) = u(\iota(a_1), \dots, \iota(a_n))$.

Definition 3.22. Let \mathbf{U} be a structure. An n -ary polymorphism u of \mathbf{U} is called *homogeneous* if for all structures $\mathbf{A} \in \text{Age}(\mathbf{U})$, for every homomorphism $f: \mathbf{A}^n \rightarrow \mathbf{U}$, for all embeddings $\iota_1, \iota_2: \mathbf{A} \hookrightarrow \mathbf{U}$ with

$$\forall (a_1, \dots, a_n) \in \mathbf{A}^n: u(\iota_1(a_1), \dots, \iota_1(a_n)) = f(a_1, \dots, a_n) = u(\iota_2(a_1), \dots, \iota_2(a_n))$$

there exists $h \in \text{Aut}(\mathbf{U})$ such that

- (1) $h \circ \iota_1 = \iota_2$,
- (2) for all $(a_1, \dots, a_n) \in \mathbf{U}^n$ we have $u(h(a_1), \dots, h(a_n)) = u(a_1, \dots, a_n)$.

Lemma 3.23. Let \mathbf{U} be a relational structure that has an n -ary universal homogeneous polymorphism u . Let $A \subseteq U$ be finite. Let f, g be n -ary polymorphisms of \mathbf{U} that agree on A^n . Then there exist selfembeddings ι_1 and ι_2 , such that

- (1) $f(x_1, \dots, x_n) = u(\iota_1(x_1), \dots, \iota_1(x_n))$,
- (2) $g(x_1, \dots, x_n) = u(\iota_2(x_1), \dots, \iota_2(x_n))$,
- (3) $\iota_1 \upharpoonright_A = \iota_2 \upharpoonright_A$.

Proof. Since u is universal, there exist $\iota_1, \iota_2: \mathbf{U} \hookrightarrow \mathbf{U}$, such that for all $(x_1, \dots, x_n) \in U^n$ we have

$$\begin{aligned} f(x_1, \dots, x_n) &= u(\iota_1(x_1), \dots, \iota_1(x_n)) \\ g(x_1, \dots, x_n) &= u(\iota_2(x_1), \dots, \iota_2(x_n)) \end{aligned}$$

Let $\hat{\iota}_i := \iota_i \upharpoonright_A$, for $i \in \{1, 2\}$, and let $\hat{f} := f \upharpoonright_{A^n}$. Let $(a_1, \dots, a_n) \in A^n$. Then we compute

$$\hat{f}(a_1, \dots, a_n) = f(a_1, \dots, a_n) = u(\iota_1(a_1), \dots, \iota_1(a_n)) = u(\hat{\iota}_1(a_1), \dots, \hat{\iota}_1(a_n)).$$

Moreover,

$$\hat{f}(a_1, \dots, a_n) = f(a_1, \dots, a_n) = g(a_1, \dots, a_n) = u(\iota_2(a_1), \dots, \iota_2(a_n)) = u(\hat{\iota}_2(a_1), \dots, \hat{\iota}_2(a_n)).$$

Since u is homogeneous, there exists an automorphism h of \mathbf{U} , such that $h \circ \hat{\iota}_1 = \hat{\iota}_2$, and such that for all $(a_1, \dots, a_n) \in U^n$ we have

$$u(h(a_1), \dots, h(a_n)) = u(a_1, \dots, a_n).$$

Let $\tilde{\iota}_1 := h \circ \iota_1$. Then $\tilde{\iota}_1 \upharpoonright_A = h \circ \hat{\iota}_1 = \hat{\iota}_2 = \iota_2 \upharpoonright_A$. Moreover, for all $(a_1, \dots, a_n) \in U^n$, we have

$$u(\tilde{\iota}_1(a_1), \dots, \tilde{\iota}_1(a_n)) = u(h(\iota_1(a_1)), \dots, h(\iota_1(a_n))) = u(\iota_1(a_1), \dots, \iota_1(a_n)) = f(a_1, \dots, a_n).$$

□

Proposition 3.24. *Let \mathbf{U} be a countably infinite relational structure that has an n -ary universal homogeneous polymorphism u . Let $(f_j)_{j < \omega}$ be a sequence of n -ary polymorphisms of \mathbf{U} that converge to an n -ary polymorphism f of \mathbf{U} . Then there is a sequence $(\iota_j)_{j < \omega}$ of selfembeddings of \mathbf{U} , and a selfembedding ι of \mathbf{U} , such that*

(1) *for every $j < \omega$ and for all $(x_1, \dots, x_n) \in U^n$ we have*

$$f_j(x_1, \dots, x_n) = u(\iota_j(x_1), \dots, \iota_j(x_n)),$$

(2) *$(\iota_j)_{j < \omega}$ converges to ι ,*

(3) *for all $(x_1, \dots, x_n) \in U^n$ we have $f(x_1, \dots, x_n) = u(\iota(x_1), \dots, \iota(x_n))$.*

Proof. Since u is universal, there exists a selfembedding ι of \mathbf{U} such that for every $(x_1, \dots, x_n) \in U^n$ we have $f(x_1, \dots, x_n) = u(\iota(x_1), \dots, \iota(x_n))$.

Suppose that the ultrametric d_U on $\mathfrak{D}_U^{(1)}$ is induced by the enumeration $(u_i)_{i < \omega}$ of U , and that d_U on $\mathfrak{D}_U^{(n)}$ is induced by the enumeration $(\bar{v}_i)_{i < \omega}$ of U^n . For every finite subset A of U let m_A be the smallest element of ω such that $A^n \subseteq \{\bar{v}_0, \dots, \bar{v}_{m_A-1}\}$. For every $i < \omega$, let $A_i := \{u_0, \dots, u_{i-1}\}$. Then $\bigcup_i A_i^n = U^n$ and thus the sequence $(m_{A_i})_{i < \omega}$ is monotonous and unbounded.

Since $(f_j)_{j < \omega}$ converges to f , for every $i < \omega$ there exists a $j_i < \omega$ such that for every $k > j_i$ we have that $D_U^{(n)}(f_k, f) > m_{A_i}$. Without loss of generality we may assume that j_i is chosen as small as possible.

For $0 \leq k < j_0$, using the fact that u is universal, we choose ι_k , such that for all $(x_1, \dots, x_n) \in U^n$

$$f_k(x_1, \dots, x_n) = u(\iota_k(x_1), \dots, \iota_k(x_n)).$$

For $j_i \leq k < j_{i+1}$, using Lemma 3.23, we chose ι_k , such that for all $(x_1, \dots, x_n) \in U^n$

$$f_k(x_1, \dots, x_n) = u(\iota_k(x_1), \dots, \iota_k(x_n)).$$

and such that ι_k agrees with ι on A_i .

It remains to observe that, the sequence $(\iota_j)_{j < \omega}$ converges to ι . Let $\varepsilon > 0$ and let

$$N := \max(-\lfloor \log_2(\varepsilon) \rfloor, 1).$$

Then, by construction, for all $k \geq j_N$, we have that ι_k agrees with ι on $\{u_0, \dots, u_{N-1}\}$ —in particular, $D_U^{(n)}(\iota_k, \iota) \geq N$, and thus $d_U(\iota_k, \iota) \leq \varepsilon$. □

Proposition 3.25. *If \mathbf{U} is a relational structure that has a k -ary universal homogeneous polymorphism u_k for every $k \in \mathbb{N} \setminus \{0\}$, then $\text{Pol}(\mathbf{U})$ has a strong gate covering.*

Proof. This is a direct consequence of Proposition 3.24, taking the set $\mathcal{U} = \{\text{Pol}^{(k)}(\mathbf{U}) \mid k \in \mathbb{N} \setminus \{0\}\}$ as an open covering of $\text{Pol}(\mathbf{U})$, and for $U = \text{Pol}^{(k)}(\mathbf{U})$ putting $f_U := u_k$. \square

4. EXISTENCE OF UNIVERSAL HOMOGENEOUS POLYMORPHISMS

Above, we saw, how the existence of universal homogeneous polymorphisms leads to the existence of strong gate coverings. In this section we derive necessary and sufficient conditions for a relational structure to have universal homogeneous polymorphisms.

In order to achieve this goal, we will make use of axiomatic Fraïssé theory as it was introduced by Droste and Göbel in [14]. As this theory is not yet in the folklore, we will recall its most important features.

4.1. Universal homogeneous objects in categories.

Definition 4.1. Let \mathcal{C} be a category in which all morphisms are monomorphisms, and let \mathcal{C}^* be a full subcategory of \mathcal{C} . An object U of \mathcal{C} is called

\mathcal{C} -universal: if for every $A \in \mathcal{C}$ there is a morphism $f: A \rightarrow U$,

\mathcal{C}^* -homogeneous: if for every $A \in \mathcal{C}^*$ and for all $f, g: A \rightarrow U$ there exists an automorphism h of U such that $h \circ f = g$,

\mathcal{C}^* -saturated: if for every $A, B \in \mathcal{C}^*$ and for all $f: A \rightarrow U, g: A \rightarrow B$ there exists some $h: B \rightarrow U$ such that $h \circ g = f$.

Example 4.2. Let \mathbf{U} be a countably infinite relational structure. Consider the category \mathcal{C} with objects

$$\{f: \mathbf{A}^n \rightarrow \mathbf{U} \mid \mathbf{A} \in \overline{\text{Age}(\mathbf{U})}\}.$$

For objects $f: \mathbf{A}^n \rightarrow \mathbf{U}$ and $g: \mathbf{B}^n \rightarrow \mathbf{U}$ the morphisms in \mathcal{C} from f to g are embeddings $\iota: \mathbf{A} \hookrightarrow \mathbf{B}$, with the property that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{B}^n & \xrightarrow{g} & \mathbf{U} \\ \uparrow \iota^n & \nearrow f & \\ \mathbf{A}^n & & \end{array}$$

In other words, for every $(a_1, \dots, a_n) \in \mathbf{A}^n$ we have $f(a_1, \dots, a_n) = g(\iota(a_1), \dots, \iota(a_n))$. Let \mathcal{C}^* be the full subcategory of \mathcal{C} that is spanned by $\{f: \mathbf{A}^n \rightarrow \mathbf{U} \mid \mathbf{A} \in \text{Age}(\mathbf{U})\}$. Now we have that a homomorphism $h: \mathbf{U}^n \rightarrow \mathbf{U}$ of \mathbf{U} is \mathcal{C} -universal if and only if h is an n -ary universal polymorphism of \mathbf{U} . Moreover, h is an n -ary homogeneous polymorphism of \mathbf{U} if and only if h is \mathcal{C}^* -homogeneous.

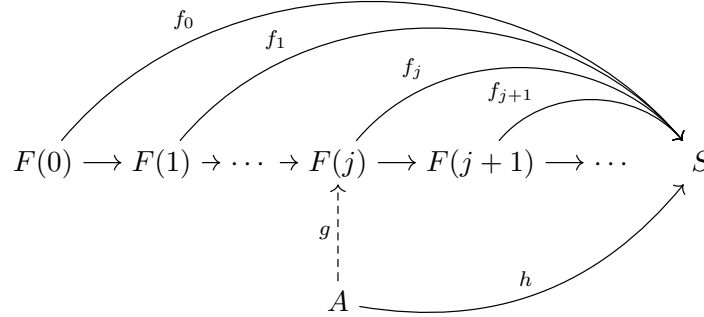
Be aware that \mathcal{C} may contain a \mathcal{C} -universal, \mathcal{C}^* -homogeneous object $u: \mathbf{V}^n \rightarrow \mathbf{U}$, but that \mathbf{V} is not isomorphic to \mathbf{U} . In the sequel it is going to be our task to give conditions on \mathcal{C} to have universal homogeneous objects and to give conditions, when there is one such object whose domain is equal to \mathbf{U}^n .

4.2. The Droste-Göbel Theorem.

Definition 4.3. Let \mathcal{C} be a category and let λ be an ordinal number. Then (λ, \leq) can be considered as a category in the usual way. The functors from (λ, \leq) to \mathcal{C} are called λ -chains of \mathcal{C} .

Definition 4.4. Let \mathcal{C} be a category and let λ be a regular cardinal number. An object A of \mathcal{C} is called λ -small if for every λ -chain $F: (\lambda, \leq) \rightarrow \mathcal{C}$ with limiting cocone $(S, (f_i)_{i < \lambda})$ and for every morphism

$h: A \rightarrow S$ there exists a $j < \lambda$ and a $g: A \rightarrow F(j)$, such that $h = f_j \circ g$.



The full subcategory of \mathcal{C} , spanned by all λ -small objects, will be denoted by $\mathcal{C}_{<\lambda}$.

Remark. Mark the similarity of the definition of λ -small elements in categories with the definition of compact elements in dcpos. Indeed, every dcpo can be considered as a category in a canonical way. With this identification, the compact elements in an ω -algebraic dcpo are just the ω -small objects of the corresponding category.

Definition 4.5. A category \mathcal{C} is called *semi- λ -algebroidal*, if:

- (1) all μ -chains ($\mu \leq \lambda$) in $\mathcal{C}_{<\lambda}$ have a colimit in \mathcal{C} .
- (2) every object in \mathcal{C} is the colimit of a λ -chain in $\mathcal{C}_{<\lambda}$.

It is called *λ -algebroidal*, if in addition $\mathcal{C}_{<\lambda}$ has up to isomorphism at most λ objects and between any two objects of $\mathcal{C}_{<\lambda}$ there are at most λ morphisms.

Remark. Mark the similarity of the definition of algebraic domains with the definition of λ -algebroidal categories. Indeed, every ω -algebraic domain, considered as a category, is ω -algebroidal.

Example 4.6. Let λ be a regular cardinal.

- (1) The category of sets of cardinality $\leq \lambda$ with injective functions is λ -algebroidal. The λ -small sets are the sets of cardinality less than λ .
- (2) If \mathbf{A} is a countably infinite structure then $(\overline{\text{Age}(\mathbf{A})}, \hookrightarrow)$ is an ω -algebroidal category. The ω -small objects in this category are the elements of $\text{Age}(\mathbf{A})$.
- (3) Groups (considered as categories with just one object) are λ -algebroidal.

Definition 4.7. Let \mathcal{C} be a category in which all morphisms are monomorphisms, and let \mathcal{C}^* be a full subcategory of \mathcal{C} . We say that

\mathcal{C}^* **has the joint embedding property:** if for all $A, B \in \mathcal{C}^*$ there exists a $C \in \mathcal{C}^*$ and morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$,

\mathcal{C}^* **has the amalgamation property:** if for all A, B, C from \mathcal{C}^* and $f: A \rightarrow B, g: A \rightarrow C$, there exists $D \in \mathcal{C}^*$ and $\hat{f}: C \rightarrow D, \hat{g}: B \rightarrow D$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \hat{g} \\ C & \xrightarrow{\hat{f}} & D \end{array}$$

Lemma 4.8. *Let \mathcal{C} be a category that has the amalgamation property and that contains a weakly initial object. Then \mathcal{C} has also the joint embedding property.*

Proof. This is clear. □

Theorem 4.9 (Droste/Göbel [14, Theorem 1.1]). *Let λ be a regular cardinal, and let \mathcal{C} be a λ -algebroidal category in which all morphisms are monomorphisms. Then, up to isomorphism, \mathcal{C} contains at most one \mathcal{C} -universal, $\mathcal{C}_{<\lambda}$ -homogeneous object. Moreover, \mathcal{C} contains a \mathcal{C} -universal, $\mathcal{C}_{<\lambda}$ -homogeneous object if and only if $\mathcal{C}_{<\lambda}$ has the joint embedding property and the amalgamation property.*

Proposition 4.10 ([14, Proposition 2.2]). *Let λ be a cardinal and let \mathcal{C} be a semi- λ -algebroidal category in which all morphisms are monic. Then for any object U of \mathcal{C} the following are equivalent:*

- (1) U is \mathcal{C} -universal and $\mathcal{C}_{<\lambda}$ -homogeneous,
- (2) U is $\mathcal{C}_{<\lambda}$ -universal and $\mathcal{C}_{<\lambda}$ -homogeneous,
- (3) U is $\mathcal{C}_{<\lambda}$ -universal and $\mathcal{C}_{<\lambda}$ -saturated.

Moreover, any two \mathcal{C} -universal, $\mathcal{C}_{<\lambda}$ -homogeneous objects in \mathcal{C} are isomorphic. Finally, if $\mathcal{C}_{<\lambda}$ contains a weakly initial object, then every $\mathcal{C}_{<\lambda}$ -saturated object is $\mathcal{C}_{<\lambda}$ -universal.

4.3. Universal homogeneous objects in comma categories.

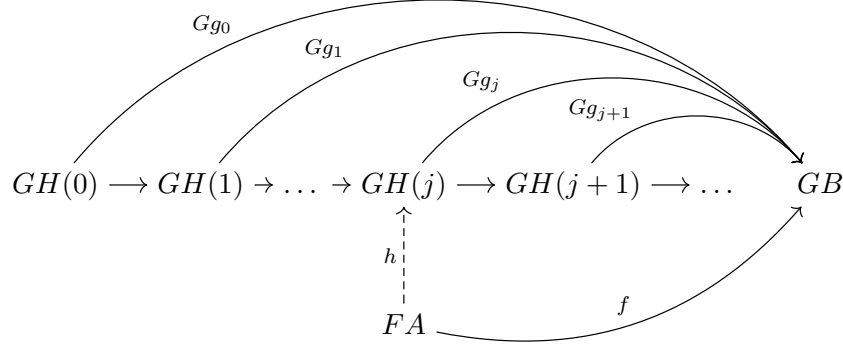
Definition 4.11. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories, let $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors. The comma category $(F \downarrow G)$ has as objects triples (A, f, B) where $A \in \mathcal{A}$, $B \in \mathcal{B}$, $f: FA \rightarrow GB$. The morphisms from (A, f, B) to (A', f', B') are pairs (a, b) such that $a: A \rightarrow A'$ in \mathcal{A} , and $b: B \rightarrow B'$ in \mathcal{B} , such that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{f} & GB \\ Fa \downarrow & & \downarrow Gb \\ FA' & \xrightarrow{f'} & GB' \end{array}$$

Definition 4.12. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories, $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors. We say that (F, G) has property

- (P1) if \mathcal{A} and \mathcal{B} are λ -algebroidal,
- (P2) if all morphisms of \mathcal{A} and \mathcal{B} are monomorphisms,
- (P3) if F preserves colimits of λ -chains,
- (P4) if $\forall \mu < \lambda$: F preserves colimits of μ -chains of λ -small objects in \mathcal{A} ,
- (P5) if G preserves colimits of λ -chains of λ -small objects in \mathcal{B} ,
- (P6) if G preserves monomorphisms,

(P7) if whenever H is a λ -chain in \mathcal{B} with limiting cocone $(B, (g_i)_{i < \lambda})$, and $A \in \mathcal{A}_{< \lambda}$, then for every $f: FA \rightarrow GB$ there exists a $j < \lambda$ and an $h: FA \rightarrow GH(j)$, such that $Gg_j \circ h = f$.



(P8) if for all $A \in \mathcal{A}_{< \lambda}$, $B \in \mathcal{B}_{< \lambda}$ there are at most λ morphisms between FA and GB in \mathcal{C} .

Proposition 4.13 ([33, Propositions 2.15, 2.16]). *Let \mathcal{A} , \mathcal{B} , \mathcal{C} be categories and let $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors. If (F, G) has properties (P1)–(P7), then $(F \downarrow G)$ is semi- λ -algebroidal. In this case, an object (A, a, B) of $(F \downarrow G)$ is λ -small if and only if $A \in \mathcal{A}_{< \lambda}$ and $B \in \mathcal{B}_{< \lambda}$. If in addition (F, G) has property (P8), then $(F \downarrow G)$ is λ -algebroidal.*

Lemma 4.14. *Let $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors such that \mathcal{B} consists just of one object and such that all morphisms of \mathcal{B} are isomorphisms. Then (F, G) has properties (P5), (P6), and (P7).*

Proof. About (P6): In categories, every isomorphism is a monomorphism, and every functor preserves isomorphisms. Hence, since every morphism of \mathcal{B} is an isomorphism, G preserves monomorphisms.

About (P7): Let $H: (\lambda, \leq) \rightarrow \mathcal{B}$ be a λ -chain with limiting cocone $(B, (g_i)_{i < \lambda})$ and let $A \in \mathcal{A}_{< \lambda}$. Moreover, let $f: FA \rightarrow GB$. For an arbitrary $j < \lambda$ define $h = Gg_j^{-1} \circ f$. Then we have $Gg_j \circ h = f$.

About (P5): Let $H: (\lambda, \leq) \rightarrow \mathcal{B}$ be a λ -chain with limiting cocone $(B, (g_i)_{i < \lambda})$ and let $(C, (c_i)_{i < \lambda})$ be a compatible cocone of $G \circ H$. Any mediating morphism $k: GB \rightarrow C$ between $(GB, (Gg_i)_{i < \lambda})$ and $(C, (c_i)_{i < \lambda})$ has to fulfill the identities $k \circ Gg_j = c_j$ for all $j \in \lambda$. It follows that the only possibility to define k is $k := c_0 \circ Gg_0^{-1}$. With this choice we compute

$$\begin{aligned} k \circ Gg_j &= c_0 \circ Gg_0^{-1} \circ Gg_j = c_0 \circ (Gg_j \circ GH(0, j))^{-1} \circ Gg_j = c_0 \circ GH(0, j)^{-1} \circ Gg_0^{-1} \circ Gg_j \\ &= c_0 \circ GH(0, j)^{-1} = c_j \circ GH(0, j) \circ GH(0, j)^{-1} = c_j. \end{aligned}$$

Thus, $(GB, (Gg_i)_{i < \lambda})$ is a limiting cocone of $G \circ H$. \square

Definition 4.15. Let \mathcal{A} , \mathcal{B} , \mathcal{C} be categories, $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors. We say that (F, G) has property

(P9) if for all $(B_1, h_1, T), (B_2, h_2, T) \in (F \downarrow G)_{< \lambda}$ there exists a $(C, h, T') \in (F \downarrow G)_{< \lambda}$ and morphisms $(f_1, g_1): (B_1, h_1, T) \rightarrow (C, h, T')$, $(f_2, g_2): (B_2, h_2, T) \rightarrow (C, h, T')$ such that the following diagram commutes:

$$\begin{array}{ccccc} FB_1 & \xrightarrow{Ff_1} & FC & \xleftarrow{Ff_2} & FB_2 \\ h_1 \downarrow & & h \downarrow & & \downarrow h_2 \\ GT & \xrightarrow{Gg_1} & GT' & \xleftarrow{Gg_2} & GT. \end{array}$$

(P10) if for all $A, B_1, B_2 \in \mathcal{A}_{<\lambda}$, $f_1: A \rightarrow B_1$, $f_2: A \rightarrow B_2$, $T \in \mathcal{B}_{<\lambda}$, $h_1: FB_1 \rightarrow GT$, $h_2: FB_2 \rightarrow GT$ with $h_1 \circ Ff_1 = h_2 \circ Ff_2$ there exist $C \in \mathcal{A}_{<\lambda}$, $T' \in \mathcal{B}_{<\lambda}$, $g_1: B_1 \rightarrow C$, $g_2: B_2 \rightarrow C$, $h: FC \rightarrow GT'$, $k: T \rightarrow T'$ such that the following diagrams commute:

$$\begin{array}{ccc}
 B_1 & \xrightarrow{g_1} & C \\
 f_1 \uparrow & & \uparrow g_2 \\
 A & \xrightarrow{f_2} & B_2
 \end{array}
 \qquad
 \begin{array}{ccc}
 FB_1 & \xrightarrow{Fg_1} & FC \\
 Ff_1 \uparrow & & \uparrow Fg_2 \\
 FA & \xrightarrow{Ff_2} & FB_2
 \end{array}$$

h_1 (curved arrow from FB_1 to GT)
 h_2 (curved arrow from FB_2 to GT)
 h (dashed curved arrow from FC to GT')
 Gk (dashed arrow from GT to GT')

(P11) if for all $A, B \in \mathcal{A}_{<\lambda}$, $T_1 \in \mathcal{B}_{<\lambda}$, $g: A \rightarrow B$, $a: FA \rightarrow GT_1$ there exist $T_2 \in \mathcal{B}_{<\lambda}$, $h: T_1 \rightarrow T_2$, $b: FB \rightarrow GT_2$ such that the following diagram commutes:

$$\begin{array}{ccc}
 FA & \xrightarrow{a} & GT_1 \\
 Fg \downarrow & & \downarrow Gh \\
 FB & \xrightarrow{b} & GT_2
 \end{array}$$

Proposition 4.16 ([33, Theorem 2.20]). *Let $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{A} \rightarrow \mathcal{C}$ be functors. Suppose that (F, G) fulfills conditions (P1)–(P8). Then the following are true:*

- (1) *If $\mathcal{B}_{<\lambda}$ has the JEP, then $(F \downarrow G)_{<\lambda}$ has the JEP if and only if (F, G) has property (P9).*
- (2) *If $\mathcal{B}_{<\lambda}$ has the AP, then $(F \downarrow G)_{<\lambda}$ has the AP if and only if (F, G) has property (P10).*

Proposition 4.17 ([33, Proposition 2.24]). *Let $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors such that (F, G) fulfills conditions (P1)–(P8). Additionally, suppose that F is faithful, and that $(F \downarrow G)_{<\lambda}$ has the JEP and the AP. Let (U, u, T) be an $(F \downarrow G)$ -universal, $(F \downarrow G)_{<\lambda}$ -homogeneous object in $(F \downarrow G)$. Then U is $\mathcal{A}_{<\lambda}$ -saturated if and only if (F, G) fulfills condition (P11).*

Proposition 4.18. *Let $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors such that (F, G) fulfills conditions (P1)–(P7). Suppose that \mathcal{B} has a $\mathcal{B}_{<\lambda}$ -universal object V . Let \mathcal{V} be a λ -algebroidal subcategory of \mathcal{B} that has V as the only object and let $J: \mathcal{V} \rightarrow \mathcal{B}$ be the identical embedding functor. Then (F, G) fulfills condition (P10) if $(F, G \circ J)$ does. Moreover, if V is $\mathcal{B}_{<\lambda}$ -saturated and (F, G) fulfills condition (P10), then so does $(F, G \circ J)$.*

Proof. Suppose, $(F, G \circ J)$ fulfills condition (P10). Given $A, B_1, B_2 \in \mathcal{A}_{<\lambda}$, $V' \in \mathcal{B}_{<\lambda}$, and morphisms h_1, h_2, f_1, f_2 that make the following diagram commutative:

$$\begin{array}{ccc}
 & & GV' \\
 & \nearrow h_1 & \\
 FB_1 & & \\
 Ff_1 \uparrow & & \\
 FA & \xrightarrow{Ff_2} & FB_2
 \end{array}$$

h_2 (curved arrow from FB_2 to GV')

Since V is $\mathcal{B}_{<\lambda}$ -universal, there exists $\iota: V' \rightarrow V$. Since $(F, G \circ J)$ fulfills condition (P10), there exist $C \in \mathcal{A}_{<\lambda}$ and morphisms g_1, g_2, h, k such that the following diagram commutes:

$$(2) \quad \begin{array}{ccccc} & & & & GV \\ & & & \nearrow^{Gk} & \\ & & & h & \\ & & & \nearrow & \\ & & & GV & \\ & \nearrow^{G\iota \circ h_1} & & \nearrow^{G\iota \circ h_2} & \\ FB_1 & \xrightarrow{Fg_1} & FC & & \\ \uparrow Ff_1 & & \uparrow Fg_2 & & \\ FA & \xrightarrow{Ff_2} & FB_2 & & \end{array}$$

and such that $g_1 \circ f_1 = g_2 \circ f_2$.

Since \mathcal{B} is λ -algebroidal, there exists a λ -chain $H: (\lambda, \leq) \rightarrow \mathcal{B}$ of λ -small objects in \mathcal{B} and morphisms $\kappa_i: Hi \rightarrow V$ ($i < \lambda$), such that $(V, (\kappa_i)_{i < \lambda})$ is a limiting cocone of H . Since $V' \in \mathcal{B}_{<\lambda}$, and $\iota: V' \rightarrow V$, there exists $j_1 < \lambda$ and $\tilde{\iota}: V' \rightarrow H_{j_1}$ such that $\iota = \kappa_{j_1} \circ \tilde{\iota}$. Moreover, since $k \circ \iota: V' \rightarrow V$, there exists $j_2 < \lambda$ and $\tilde{k}: V' \rightarrow H_{j_2}$ such that $k \circ \iota = \kappa_{j_2} \circ \tilde{k}$.

Since $C \in \mathcal{A}_{<\lambda}$, $h: FC \rightarrow GV$, and since (F, G) fulfills condition (P7), there exists $j_3 < \lambda$, $\tilde{h}: FC \rightarrow GH_{j_3}$ such that $h = G\kappa_{j_3} \circ \tilde{h}$. Let j be the maximum of $\{j_1, j_2, j_3\}$. Then we have

$$(3) \quad \iota = \kappa_j \circ H(j_1, j) \circ \tilde{\iota}$$

$$(4) \quad k \circ \iota = \kappa_j \circ H(j_2, j) \circ \tilde{k}$$

$$(5) \quad h = G\kappa_j \circ GH(j_3, j) \circ \tilde{h}$$

Let us define

$$(6) \quad \hat{k} := H(j_2, j) \circ \tilde{k},$$

$$(7) \quad \hat{h} := GH(j_3, j) \circ \tilde{h}$$

It remains to show that the following diagram commutes:

$$\begin{array}{ccccc} & & & & GHj \\ & & & \nearrow^{G\hat{k}} & \\ & & & \hat{h} & \\ & & & \nearrow & \\ & & & GV' & \\ & \nearrow^{h_1} & & \nearrow^{h_2} & \\ FB_1 & \xrightarrow{Fg_1} & FC & & \\ \uparrow Ff_1 & & \uparrow Fg_2 & & \\ FA & \xrightarrow{Ff_2} & FB_2 & & \end{array}$$

For this we calculate

$$\begin{aligned} G\kappa_j \circ G\hat{k} \circ h_1 &\stackrel{(5)}{=} G(\kappa_j \circ H(j_2, j) \circ \tilde{k}) \circ h_1 \stackrel{(3)}{=} Gk \circ G\iota \circ h_1 \stackrel{(2)}{=} h \circ Fg_1 \\ &\stackrel{(4)}{=} G\kappa_j \circ GH(j_3, j) \circ \tilde{h} \circ Fg_1 \stackrel{(6)}{=} G\kappa_j \circ \hat{h} \circ Fg_1 \end{aligned}$$

Since κ_j is a monomorphism and since G preserves monos, we conclude $G\hat{k} \circ h_1 = \hat{h} \circ Fg_1$. Analogously one shows $G\hat{k} \circ h_2 = \hat{h} \circ Fg_2$. Thus we showed that (F, G) fulfills condition (P10).

Suppose now that V is $\mathcal{B}_{<\lambda}$ -saturated and that (F, G) fulfills condition (P10). Let $A, B_1, B_2 \in \mathcal{A}_{<\lambda}$ and let f_1, f_2, h_1, h_2 be morphisms such that the following diagram commutes:

$$\begin{array}{ccc} & & GV \\ & \nearrow^{h_1} & \uparrow^{h_2} \\ FB_1 & & \\ \uparrow^{Ff_1} & & \\ FA & \xrightarrow{Ff_2} & FB_2. \end{array}$$

Since \mathcal{B} is λ -algebroidal, there exists a λ -chain $H: (\lambda, \leq) \rightarrow \mathcal{B}$ of λ -small objects of \mathcal{B} and morphisms $v_i: Hi \rightarrow V$ ($i < \lambda$) such that $(V, (v_i)_{i < \lambda})$ is a limiting cocone of H . By condition (P7), there exist $j_1, j_2 < \lambda$, $\tilde{h}_1: FB_1 \rightarrow GHj_1$, $\tilde{h}_2: FB_2 \rightarrow GHj_2$, such that $h_1 = Gv_{j_1} \circ \tilde{h}_1$, $h_2 = Gv_{j_2} \circ \tilde{h}_2$. Let j be the maximum of $\{j_1, j_2\}$. Then

$$\begin{aligned} (7) \quad h_1 &= Gv_j \circ GH(j_1, j) \circ \tilde{h}_1, \text{ and} \\ h_2 &= Gv_j \circ GH(j_2, j) \circ \tilde{h}_2. \end{aligned}$$

Let

$$\begin{aligned} (8) \quad \hat{h}_1 &:= H(j_1, j) \circ \tilde{h}_1, \text{ and let} \\ \hat{h}_2 &:= H(j_2, j) \circ \tilde{h}_2. \end{aligned}$$

Since (F, G) fulfills condition (P10), there exist $C \in \mathcal{A}_{<\lambda}$, $V' \in \mathcal{B}_{<\lambda}$, and morphisms $g_1, g_2, \hat{h}, \hat{k}$ such that the following diagram commutes:

$$(9) \quad \begin{array}{ccccc} & & & & GV' \\ & & & \nearrow^{G\hat{k}} & \\ & & GHj & \nearrow^{\hat{h}} & \\ & \nearrow^{\hat{h}_1} & & & \\ FB_1 & \xrightarrow{Fg_1} & FC & \xrightarrow{\quad} & \\ \uparrow^{Ff_1} & & \uparrow^{Fg_2} & & \\ FA & \xrightarrow{Ff_2} & FB_2 & \nearrow^{\hat{h}_2} & \end{array}$$

Since V is $\mathcal{B}_{<\lambda}$ -saturated and since $v_j: Hj \rightarrow V$ and $\hat{k}: Hj \rightarrow V'$, there exists $\hat{v}_j: V' \rightarrow V$ such that

$$(10) \quad v_j = \hat{v}_j \circ \hat{k}.$$

It remains to show that the following diagram commutes:

$$\begin{array}{ccc}
 & & GV \\
 & \nearrow^{h_1} & \\
 FB_1 & \xrightarrow{Fg_1} & FC \\
 \uparrow Ff_1 & & \uparrow Fg_2 \\
 FA & \xrightarrow{Ff_2} & FB_2
 \end{array}
 \quad
 \begin{array}{c}
 \\
 \\
 \\
 \nearrow^{G\hat{v}_j \circ \hat{h}} \\
 \\
 \searrow_{h_2}
 \end{array}$$

To this end we calculate:

$$G\hat{v}_j \circ \hat{h} \circ Fg_1 \stackrel{(9)}{=} G\hat{v}_j \circ G\hat{k} \circ \hat{h}_1 \stackrel{(10)}{=} Gv_j \circ \hat{h}_1 \stackrel{(8)}{=} Gv_j H(j_1, j) \circ \tilde{h}_1 \stackrel{(7)}{=} h_1.$$

Analogously one shows that $G\hat{v}_j \circ \hat{h} \circ Fg_2 = h_2$. Thus, $(F, G \circ J)$ fulfills condition (P10). \square

Proposition 4.19. *Let $F: \mathcal{A} \rightarrow \mathcal{C}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be functors such that (F, G) fulfills conditions (P1)–(P7). Suppose that \mathcal{B} has a $\mathcal{B}_{<\lambda}$ -universal object V . Let \mathcal{V} be a subcategory of \mathcal{B} that has V as the only object and let $J: \mathcal{V} \rightarrow \mathcal{B}$ be the identical embedding functor. Then (F, G) fulfills condition (P11) if $(F, G \circ J)$ does. Moreover, if V is $\mathcal{B}_{<\lambda}$ -saturated and if (F, G) fulfills condition (P11), then so does $(F, G \circ J)$.*

Proof. Since \mathcal{B} is λ -algebroidal, there exists a λ -chain $H: (\lambda, \leq) \rightarrow \mathcal{B}$ of λ -small objects in \mathcal{B} and morphisms $v_i: Hi \rightarrow V$ for every $i < \lambda$, such that $(V, (v_i)_{i < \lambda})$ is a limiting cocone for H .

Suppose that $(F, G \circ J)$ fulfills condition (P11). Let $A, B \in \mathcal{A}_{<\lambda}$, $T \in \mathcal{B}_{<\lambda}$, $g: A \rightarrow B$, $a: FA \rightarrow GT$. Since V is $\mathcal{B}_{<\lambda}$ -universal, there exists $\iota: T \rightarrow V$. Hence, by condition (P11), there exists $h: V \rightarrow V$, $b: FB \rightarrow GV$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 FA & \xrightarrow{a} & GT & \xrightarrow{G\iota} & GV \\
 Fg \downarrow & & & & \downarrow Gh \\
 FB & \xrightarrow{\quad b \quad} & & & GV.
 \end{array}
 \tag{11}$$

By condition (P7), there exists $j_1 < \lambda$, $\tilde{b}: FB \rightarrow GHj_1$ such that $b = Gv_{j_1} \circ \tilde{b}$. Moreover, since $T \in \mathcal{B}_{<\lambda}$, there exists $j_2 < \lambda$, $\tilde{h}: T \rightarrow Hj_2$ such that $h \circ \iota = v_{j_2} \circ \tilde{h}$. Let j be the maximum of $\{j_1, j_2\}$. Then

$$(12) \quad b = Gv_j \circ GH(j_1, j) \circ \tilde{b} \text{ and}$$

$$(13) \quad h \circ \iota = v_j \circ H(j_2, j) \circ \tilde{h}.$$

Define

$$(14) \quad \hat{b} := GH(j_1, j) \circ \tilde{b} \text{ and}$$

$$(15) \quad \hat{h} := H(j_2, j) \circ \tilde{h}.$$

It remains to observe that the following diagram commutes:

$$\begin{array}{ccc}
 FA & \xrightarrow{a} & GT \\
 Fg \downarrow & & \downarrow G\hat{h} \\
 FB & \xrightarrow{\quad \hat{b} \quad} & GHj.
 \end{array}$$

Indeed, we compute

$$\begin{aligned} Gv_j \circ G\hat{h} \circ a &\stackrel{(15)}{=} Gv_j \circ GH(j_2, j) \circ G\tilde{h} \circ a \stackrel{(13)}{=} Gh \circ G\iota \circ a \stackrel{(11)}{=} b \circ Fg \\ &\stackrel{(12)}{=} Gv_j \circ GH(j_1, j) \circ \tilde{b} \circ Fg \stackrel{(14)}{=} Gv_j \circ \hat{b} \circ Fg. \end{aligned}$$

Since v_j is a monomorphism and since G preserves monos, we obtain $G\hat{h} \circ a = \hat{b} \circ Fg$. Thus (F, G) fulfills condition (P11).

Suppose now that (F, G) fulfills condition (P11) and that V is $\mathcal{B}_{<\lambda}$ -saturated. Let $A, B \in \mathcal{A}_{<\lambda}$, $g: A \rightarrow B$, and $a: FA \rightarrow GV$. Then, by condition (P7), there exists $j < \lambda$ and $\hat{b}: FA \rightarrow GHj$ such that

$$(16) \quad a = Gv_j \circ \hat{a}.$$

By condition (P11), there exists $V' \in \mathcal{B}_{<\lambda}$, $\hat{b}: FB \rightarrow GV'$, $\hat{h}: Hj \rightarrow V'$ such that the following diagram commutes:

$$(17) \quad \begin{array}{ccc} FA & \xrightarrow{\hat{a}} & GHj & \xrightarrow{Gv_j} & GV \\ Fg \downarrow & & \downarrow G\hat{h} & & \\ FB & \xrightarrow{\hat{b}} & GV' & & \end{array}$$

Since V is $\mathcal{B}_{<\lambda}$ -saturated, there exists $\iota: V' \rightarrow V$ such that

$$(18) \quad \iota \circ \hat{h} = v_j.$$

It remains to observe that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{a} & GV \\ Fg \downarrow & \nearrow G\iota \circ \hat{b} & \\ FB & & \end{array}$$

Indeed, we compute

$$G\iota \circ \hat{b} \circ Fg \stackrel{(17)}{=} G\iota \circ G\hat{h} \circ \hat{a} \stackrel{(18)}{=} Gv_j \circ \hat{a} \stackrel{(16)}{=} a.$$

Thus, $(F, G \circ J)$ fulfills (P11). □

4.4. Criteria for the existence of universal homogeneous polymorphisms. In the following we fix a signature $\underline{\Sigma}$. With $\mathcal{C}_{\underline{\Sigma}}$ we will denote the category of all $\underline{\Sigma}$ -structures with homomorphisms as morphisms. Moreover, we fix an arbitrary countably infinite $\underline{\Sigma}$ -structure \mathbf{U} , and for every $n \in \mathbb{N} \setminus \{0\}$ we denote by $P_n: (\text{Age}(\mathbf{U}), \hookrightarrow) \rightarrow \mathcal{C}_{\underline{\Sigma}}$ the functor given by

$$P_n: \mathbf{A} \mapsto \mathbf{A}^n, \quad f \mapsto f^n.$$

Finally, by \mathcal{B} we will denote the category that has only one object \mathbf{U} and only one morphism $1_{\mathbf{U}}$, and with G we will denote the identical embedding functor from \mathcal{B} to $\mathcal{C}_{\underline{\Sigma}}$.

Lemma 4.20. *With the notions from above, For every $n \in \mathbb{N} \setminus \{0\}$, the functor P_n preserves colimits of ω -chains.*

Proof. We are going to make use of the fact that we know how colimits of chains may be constructed in $(\text{Age}(\mathbf{U}), \hookrightarrow)$ and in \mathcal{C}_{Σ} .

Let $H: (\omega, \leq) \rightarrow (\text{Age}(\mathbf{U}), \hookrightarrow)$. Without loss of generality, we may assume that for all $j_1 \leq j_2 \in \omega$ we have that $Hj_1 \leq Hj_2$, and that $H(j_1, j_2): Hj_1 \hookrightarrow Hj_2$ is the identical embedding. For better readability, for every $j \in \omega$, we will denote Hj by \mathbf{V}_j .

Let $\mathbf{V} := \bigcup_{j < \omega} \mathbf{V}_j$ and let $v_j: \mathbf{V}_j \hookrightarrow \mathbf{V}$ be the identical embedding. Then $(\mathbf{V}, (v_j)_{j \in \omega})$ is a limiting cocone of H .

Note now that that for all $j_1 \leq j_2 < \omega$ we have that $P_n(H(j_1, j_2)): \mathbf{V}_{j_1}^n \hookrightarrow \mathbf{V}_{j_2}^n$ is the identical embedding and that for every $j \in \omega$ we have that $P_n(v_j): \mathbf{V}_j^n \hookrightarrow \mathbf{V}^n$ is the identical embedding, too. Moreover, $\bigcup_{j \in \omega} \mathbf{V}_j^n = \mathbf{V}^n$. Thus, $(\mathbf{V}^n, (v_j^n)_{j \in \omega})$ is a limiting cocone of $P_n \circ H$. It follows that P_n preserves colimits of ω -chains. \square

Lemma 4.21. *With the notions from above the comma-category $(P_n \downarrow G)$ is ω -algebroidal.*

Proof. We already noted above (cf. Example 4.6) that $(\text{Age}(\mathbf{U}), \hookrightarrow)$ and \mathcal{B} are ω -algebroidal. Moreover, by definition, all morphisms of \mathcal{B} and $(\text{Age}(\mathbf{U}), \hookrightarrow)$ are monomorphisms. Thus, (P_n, G) has properties (P1) and (P2). By Lemma 4.20, (P_n, G) fulfills property (P3). Trivially, P_n preserves colimits of finite chains. Thus (P_n, G) satisfies property (P4). Now, by Lemma 4.14, (P_n, G) fulfills properties (P5), (P6), (P7).

Let $\mathbf{A} \in \text{Age}(\mathbf{U})$. Then we have that $P_n(\mathbf{A}) = \mathbf{A}^n$ is finite, too. Hence, since \mathbf{U} is countable, there are just countably many homomorphisms from \mathbf{A}^n to \mathbf{U} . Thus, (P_n, G) fulfills condition (P8).

Now, by Proposition 4.13, $(P_n \downarrow G)$ is ω -algebroidal. \square

Lemma 4.22. *With the notions from above, the comma-category $(P_n \downarrow P_1)$ is ω -algebroidal.*

Proof. We already noted above that $(\text{Age}(\mathbf{U}), \hookrightarrow)$ is ω -algebroidal. Moreover, all morphisms of $(\text{Age}(\mathbf{U}), \hookrightarrow)$ are monomorphisms. Thus (P_n, P_1) has properties (P1) and (P2). By Lemma 4.20, (P_n, P_1) has properties (P3) and (P5). Trivially, P_n preserves colimits of finite chains. Thus (P_n, P_1) fulfills property (P4). Since every morphism of $(\text{Age}(\mathbf{U}), \hookrightarrow)$ is an embedding, every embedding is a monomorphism in \mathcal{C}_{Σ} , and since P_1 is the identical embedding functor, we have that (P_n, P_1) fulfills property (P6).

Since P_n maps finite structures to finite structures, and since P_1 is the identical embedding functor, (P_n, P_1) satisfies property (P7).

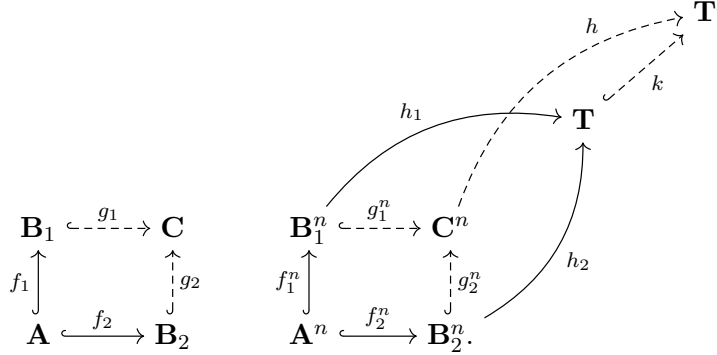
Again, since P_n maps finite structures to finite structures, (P_n, P_1) has property (P8).

Now, by Proposition 4.13, $(P_n \downarrow P_1)$ is ω -algebroidal. \square

Observation 4.23. *With the notions from above, a polymorphism $u: \mathbf{U}^n \rightarrow \mathbf{U}$ is universal and homogeneous if and only if $(\mathbf{U}, u, \mathbf{U})$ is $(P_n \downarrow G)$ -universal and $(P_n \downarrow G)_{<\omega}$ -homogeneous.*

Definition 4.24. Let \mathcal{C} be a class of structures of the same type, and let $n \in \mathbb{N} \setminus \{0\}$. We say that \mathcal{C} has the AEPⁿ if for all $\mathbf{A}, \mathbf{B}_i, \mathbf{T} \in \mathcal{C}$, $f_i: \mathbf{A} \hookrightarrow \mathbf{B}_i$, $h_i: \mathbf{B}_i^n \rightarrow \mathbf{T}$ (where $i \in \{1, 2\}$), with $h_1 \circ f_1^n = h_2 \circ f_2^n$, there exist $\mathbf{C}, \mathbf{T}' \in \mathcal{C}$, $g_i: \mathbf{B}_i \hookrightarrow \mathbf{C}$ (where $i \in \{1, 2\}$), $h: \mathbf{C}^n \rightarrow \mathbf{T}'$, $k: \mathbf{T} \hookrightarrow \mathbf{T}'$

such that the following diagrams commute:



Definition 4.25. Let \mathcal{C} be a class of structures of the same type, and let $n \in \mathbb{N} \setminus \{0\}$. We say that \mathcal{C} has the HAP^n if for all $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ $g: \mathbf{A} \hookrightarrow \mathbf{B}$, $\mathbf{T}_1 \in \mathcal{C}$, $a: \mathbf{A}^n \rightarrow \mathbf{T}_1$ there exist $\mathbf{T}_2 \in \mathcal{C}$, $b: \mathbf{B}^n \rightarrow \mathbf{T}_2$, $h: \mathbf{T}_1 \hookrightarrow \mathbf{T}_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A}^n & \xrightarrow{a} & \mathbf{T}_1 \\ g^n \downarrow & & \downarrow h \\ \mathbf{B}^n & \xrightarrow{b} & \mathbf{T}_2. \end{array}$$

If $n = 1$, then the HAP^n is just the HAP.

Remark. Note that if \mathcal{C} is closed with respect to finite products, then it has the HAP^n for every $n \in \mathbb{N} \setminus \{0\}$ if and only if it has the HAP.

Theorem 4.26. Let \mathbf{U} be a countable homogeneous relational structure and let $n \in \mathbb{N} \setminus \{0\}$. Then \mathbf{U} has an n -ary universal homogeneous polymorphism if and only if $\text{Age}(\mathbf{U})$ has the AEP^n and the HAP^n .

Proof. Consider the categories and functors from the beginning of Section 4.4. From Lemmas 4.21 and 4.22 it follows (P_n, G) , and (P_n, P_1) are both ω -algebroidal.

“ \Rightarrow ”: Suppose that $\text{Age}(\mathbf{U})$ has the AEP^n and the HAP^n . Then we have that (P_n, P_1) fulfills properties (P10) and (P11).

Note now that \mathcal{B} is an ω -algebroidal subcategory of $(\overline{\text{Age}(\mathbf{U})}, \hookrightarrow)$. Let $J: \mathcal{B} \rightarrow (\overline{\text{Age}(\mathbf{U})}, \hookrightarrow)$ be the identical embedding functor. Then $G = P_1 \circ J$. By assumption, \mathbf{U} is both, $(\overline{\text{Age}(\mathbf{U})}, \hookrightarrow)$ -universal and $(\text{Age}(\mathbf{U}), \hookrightarrow)$ -homogeneous. Thus, from Proposition 4.10 it follows that \mathbf{U} is $(\text{Age}(\mathbf{U}), \hookrightarrow)$ -saturated. Now we may conclude from Proposition 4.18, that (P_n, G) has property (P10). Clearly, $(\mathcal{B})_{<\omega}$ has the JEP and the AP. Now, from Proposition 4.16, it follows that $(P_n \downarrow G)$ has the AP. Note that $(\emptyset, \emptyset, \mathbf{U})$ is an initial object in $(P_n \downarrow G_3)_{<\omega}$. Hence, by Lemma 4.8, $(P_n \downarrow G)_{<\omega}$ has the JEP. Now, from Proposition 4.13 together with Theorem 4.9 it follows that there exists an $(P_n \downarrow G)$ -universal, $(P_n \downarrow G)_{<\omega}$ -homogeneous object $(\mathbf{V}, w, \mathbf{U})$. From Proposition 4.19 it follows that (P_n, G) has property (P11). Since P_n is faithful, from Proposition 4.17 we conclude that \mathbf{V} is $(\text{Age}(\mathbf{U}), \hookrightarrow)$ -saturated. Since \emptyset is initial in $(\text{Age}(\mathbf{U}), \hookrightarrow)$, and since all morphisms of $(\text{Age}(\mathbf{U}), \hookrightarrow)$ are monomorphisms, from Proposition 4.10 it follows that \mathbf{V} is $(\overline{\text{Age}(\mathbf{U})}, \hookrightarrow)$ -universal and $(\text{Age}(\mathbf{U}), \hookrightarrow)$ -homogeneous. In other words, \mathbf{V} is universal and homogeneous with the same age like \mathbf{U} . Thus, from Fraïssé’s Theorem, it follows that there is an isomorphism $h: \mathbf{U} \rightarrow \mathbf{V}$. Now define $u := w \circ P_n(h)$. Then $(h, 1_{\mathbf{U}}): (\mathbf{U}, u, \mathbf{U}) \rightarrow (\mathbf{V}, w, \mathbf{U})$ is an isomorphism in $(P_n \downarrow G)$. In particular,

$(\mathbf{U}, u, \mathbf{U})$ is $(P_n \downarrow G)$ -universal and $(P_n \downarrow G)_{<\omega}$ -homogeneous. By Observation 4.23, u is an n -ary universal homogeneous polymorphism of \mathbf{U} .

“ \Leftarrow ”: Suppose that \mathbf{U} has an n -ary universal homogeneous polymorphism u . Then, by Observation 4.23 $(\mathbf{U}, u, \mathbf{U})$ is $(P_n \downarrow G)$ -universal, $(P_n \downarrow G)_{<\omega}$ -homogeneous. Since $\text{Age}(\mathbf{U})$ has the AP and the JEP, it follows from Proposition 4.16 that (P_n, G) has properties (P9) and (P10). Moreover, since \mathbf{U} is homogeneous, it follows from Proposition 4.10, that it is $(\text{Age}(\mathbf{U}), \hookrightarrow)$ -saturated. Since P_n is faithful, from Proposition 4.17 it follows that (P_n, G) has property (P11).

\mathbf{U} is universal. In other words, it is $(\text{Age}(\mathbf{U}), \hookrightarrow)$ -universal. Note also that \mathcal{B} is a λ -algebroidal subcategory of $(\text{Age}(\mathbf{U}), \hookrightarrow)$. Now, from Propositions 4.18 and 4.19 it follows that (P_n, P_1) has properties (P10), (P11). However, this is the same as to say that $\text{Age}(\mathbf{U})$ has the AEP^n and the HAP^n . \square

4.5. Sufficient condition for the existence of universal homogeneous polymorphisms. Though, Theorem 4.26 gives necessary and sufficient conditions for countable homogeneous relational structures to have universal homogeneous polymorphisms, unfortunately, these conditions are relatively difficult to verify. The goal of this section is to give sufficient conditions for the existence of universal homogeneous polymorphisms, that are somewhat easier to test.

Definition 4.27. A class \mathcal{C} of Σ -structures is said to have the *strict amalgamation property* if \mathcal{C} has the amalgamation property and if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{C}$, and for all embeddings $f_1: \mathbf{A} \hookrightarrow \mathbf{B}_1$, $f_2: \mathbf{A} \hookrightarrow \mathbf{B}_2$ there exists some $\mathbf{C} \in \mathcal{C}$ and homomorphisms $g_1: \mathbf{B}_1 \rightarrow \mathbf{C}$, $g_2: \mathbf{B}_2 \rightarrow \mathbf{C}$ such that the following is a pushout-square in $(\mathcal{C}, \rightarrow)$:

$$(19) \quad \begin{array}{ccc} \mathbf{B}_1 & \xrightarrow{\quad g_1 \quad} & \mathbf{C} \\ f_1 \uparrow & & \uparrow g_2 \\ \mathbf{A} & \xrightarrow{\quad f_2 \quad} & \mathbf{B}_2. \end{array}$$

An age that has the strict amalgamation property is called a *strict Fraïssé-class*.

Remark. The homomorphisms g_1 and g_2 in equation (19) are automatically embeddings, because \mathcal{C} has the amalgamation property. If f_1, f_2, g_1, g_2 are identical embeddings, then the structure \mathbf{C} will be denoted by $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$ and will be called the *amalgamated free sum* of \mathbf{B}_1 and \mathbf{B}_2 with respect to \mathbf{A} .

Note also that every Fraïssé class that has the free amalgamation property is also a strict Fraïssé class.

Definition 4.28. Let \mathcal{C} be a class of Σ -structures closed under finite products and enjoying the strict amalgamation property. We say that \mathcal{C} has *well-behaved amalgamated free sums* if for all pushout-diagrams

$$\begin{array}{ccc} \mathbf{B}_1 & \xrightarrow{\kappa_{\mathbf{B}_1}} & \mathbf{B}_1 \oplus_{\mathbf{A}_1} \mathbf{C}_1 \\ \uparrow \cong & & \uparrow \kappa_{\mathbf{C}_1} \\ \mathbf{A}_1 & \xrightarrow{\quad = \quad} & \mathbf{C}_1 \end{array} \quad \begin{array}{ccc} \mathbf{B}_2 & \xrightarrow{\kappa_{\mathbf{B}_2}} & \mathbf{B}_2 \oplus_{\mathbf{A}_2} \mathbf{C}_2 \\ \uparrow \cong & & \uparrow \kappa_{\mathbf{C}_2} \\ \mathbf{A}_2 & \xrightarrow{\quad = \quad} & \mathbf{C}_2 \end{array}$$

$$\begin{array}{ccc} \mathbf{B}_1 \times \mathbf{B}_2 & \xrightarrow{\kappa_{\mathbf{B}_1 \times \mathbf{B}_2}} & (\mathbf{B}_1 \times \mathbf{B}_2) \oplus_{\mathbf{A}_1 \times \mathbf{A}_2} (\mathbf{C}_1 \times \mathbf{C}_2) \\ \uparrow \cong & & \uparrow \kappa_{\mathbf{C}_1 \times \mathbf{C}_2} \\ \mathbf{A}_1 \times \mathbf{A}_2 & \xrightarrow{\quad = \quad} & \mathbf{C}_1 \times \mathbf{C}_2 \end{array}$$

in $(\mathcal{C}, \rightarrow)$, the unique homomorphism $h: (\mathbf{B}_1 \times \mathbf{B}_2) \oplus_{\mathbf{A}_1 \times \mathbf{A}_2} (\mathbf{C}_1 \times \mathbf{C}_2) \rightarrow (\mathbf{B}_1 \oplus_{\mathbf{A}_1} \mathbf{C}_1) \times (\mathbf{B}_2 \oplus_{\mathbf{A}_2} \mathbf{C}_2)$ that makes the following diagram commutative

$$\begin{array}{ccc}
 & & \xrightarrow{\kappa_{\mathbf{B}_1} \times \kappa_{\mathbf{B}_2}} (\mathbf{B}_1 \oplus_{\mathbf{A}_1} \mathbf{C}_1) \times (\mathbf{B}_2 \oplus_{\mathbf{A}_2} \mathbf{C}_2) \\
 & \nearrow h & \\
 \mathbf{B}_1 \times \mathbf{B}_2 & \xrightarrow{\kappa_{\mathbf{B}_1 \times \mathbf{B}_2}} (\mathbf{B}_1 \times \mathbf{B}_2) \oplus_{\mathbf{A}_1 \times \mathbf{A}_2} (\mathbf{C}_1 \times \mathbf{C}_2) & \\
 \uparrow = & \uparrow \kappa_{\mathbf{C}_1 \times \mathbf{C}_2} & \\
 \mathbf{A}_1 \times \mathbf{A}_2 & \xrightarrow{=} \mathbf{C}_1 \times \mathbf{C}_2 & \xrightarrow{\kappa_{\mathbf{C}_1} \times \kappa_{\mathbf{C}_2}}
 \end{array}$$

is an embedding.

Lemma 4.29. *Let \mathcal{C} be a class of $\underline{\Sigma}$ -structures with the strict amalgamation property, that is closed under finite products. Suppose further that \mathcal{C} has well-behaved amalgamated free sums. Given a pushout square*

$$\begin{array}{ccc}
 \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} \\
 \uparrow = & & \uparrow g_2 \\
 \mathbf{A} & \xrightarrow{=} & \mathbf{B}_2.
 \end{array}$$

Consider the pushout square

$$\begin{array}{ccc}
 \mathbf{B}_1^n & \xrightarrow{\hat{g}_1} & \hat{\mathbf{C}} \\
 \uparrow = & & \uparrow \hat{g}_2 \\
 \mathbf{A}^n & \xrightarrow{=} & \mathbf{B}_2^n.
 \end{array}$$

Then the unique mediating morphism $k: \hat{\mathbf{C}} \rightarrow \mathbf{C}^n$, that makes the following diagram commutative:

$$\begin{array}{ccc}
 & & \xrightarrow{g_1^n} \mathbf{C}^n \\
 & \nearrow k & \\
 \mathbf{B}_1^n & \xrightarrow{\hat{g}_1} \hat{\mathbf{C}} & \\
 \uparrow = & \uparrow \hat{g}_2 & \\
 \mathbf{A}^n & \xrightarrow{=} \mathbf{B}_2^n & \xrightarrow{g_2^n}
 \end{array}$$

is an embedding.

Proof. We proceed by induction on n . The case $n = 1$ is immediate.

Consider the following pushout square:

$$\begin{array}{ccc}
 \mathbf{B}_1^n & \xrightarrow{\tilde{g}_1} & \tilde{\mathbf{C}} \\
 \uparrow = & & \uparrow \tilde{g}_2 \\
 \mathbf{A}^n & \xrightarrow{=} & \mathbf{B}_2^n.
 \end{array}$$

By induction hypothesis, the unique mediation arrow \tilde{k} in the following diagram is an embedding:

$$\begin{array}{ccc}
 & & \mathbf{C}^n \\
 & \nearrow^{g_1^n} & \\
 \mathbf{B}_1^n & \xrightarrow{\tilde{g}_1} & \tilde{\mathbf{C}} \\
 \uparrow \scriptstyle{=} & & \uparrow \scriptstyle{\tilde{g}_2} \\
 \mathbf{A}^n & \xrightarrow{=} & \mathbf{B}_2^n
 \end{array}
 \quad \begin{array}{c}
 \nwarrow \scriptstyle{\tilde{k}} \\
 \nearrow \scriptstyle{g_2^n}
 \end{array}$$

Since \mathcal{C} has well-behaved amalgamated free sums, the mediating arrow \hat{k} in the following diagram is an embedding, too:

$$\begin{array}{ccc}
 & & \tilde{\mathbf{C}} \times \mathbf{C} \\
 & \nearrow^{\tilde{g}_1 \times g_1} & \\
 \mathbf{B}_1^{n+1} & \xrightarrow{\hat{g}_1} & \hat{\mathbf{C}} \\
 \uparrow \scriptstyle{=} & & \uparrow \scriptstyle{\hat{g}_2} \\
 \mathbf{A}^{n+1} & \xrightarrow{=} & \mathbf{B}_2^{n+1}
 \end{array}
 \quad \begin{array}{c}
 \nwarrow \scriptstyle{\hat{k}} \\
 \nearrow \scriptstyle{\tilde{g}_2 \times g_2}
 \end{array}$$

We conclude that then the following diagram commutes:

$$\begin{array}{ccc}
 & & \mathbf{C}^{n+1} \\
 & \nearrow^{g_1^{n+1}} & \\
 \mathbf{B}_1^{n+1} & \xrightarrow{\hat{g}_1} & \hat{\mathbf{C}} \\
 \uparrow \scriptstyle{=} & & \uparrow \scriptstyle{\hat{g}_2} \\
 \mathbf{A}^{n+1} & \xrightarrow{=} & \mathbf{B}_2^{n+1}
 \end{array}
 \quad \begin{array}{c}
 \nwarrow \scriptstyle{\tilde{g}_1 \times g_1} \\
 \nearrow \scriptstyle{\tilde{g}_2 \times g_2} \\
 \nwarrow \scriptstyle{\tilde{k} \times 1_{\mathbf{C}}}
 \end{array}$$

Hence $k := (\tilde{k} \times 1_{\mathbf{C}}) \circ \hat{k}$ is the unique mediating morphism that makes the following diagram commutative:

$$\begin{array}{ccc}
 & & \mathbf{C}^{n+1} \\
 & \nearrow^{g_1^{n+1}} & \\
 \mathbf{B}_1^{n+1} & \xrightarrow{\hat{g}_1} & \hat{\mathbf{C}} \\
 \uparrow \scriptstyle{=} & & \uparrow \scriptstyle{\hat{g}_2} \\
 \mathbf{A}^{n+1} & \xrightarrow{=} & \mathbf{B}_2^{n+1}
 \end{array}
 \quad \begin{array}{c}
 \nwarrow \scriptstyle{k} \\
 \nearrow \scriptstyle{g_2^{n+1}}
 \end{array}$$

Moreover, since both, $\tilde{k} \times 1_{\mathbf{C}}$ and \hat{k} are embeddings, we have that k is an embedding, too. \square

Proposition 4.30. *Let \mathcal{C} be a class of $\underline{\Sigma}$ -structures such that*

- (1) \mathcal{C} has the strict amalgamation property,
- (2) \mathcal{C} is closed with respect to finite products,
- (3) \mathcal{C} has well-behaved amalgamated free sums,
- (4) \mathcal{C} has the HAP.

Then \mathcal{C} has the AEPⁿ, for every $n \in \mathbb{N} \setminus \{0\}$.

Proof. Let $\mathbf{A}, \mathbf{B}_i, \mathbf{T} \in \mathcal{C}$, $f_i: \mathbf{A} \hookrightarrow \mathbf{B}_i$, $h_i: \mathbf{B}_i^n \rightarrow \mathbf{T}$ (where $i \in \{1, 2\}$), with $h_1 \circ f_1^n = h_2 \circ f_2^n$.

Let $\mathbf{C} \in \mathcal{C}$, $g_1: \mathbf{B}_1 \hookrightarrow \mathbf{C}$, $g_2: \mathbf{B}_2 \hookrightarrow \mathbf{C}$ such that the following is a pushout-square in $(\mathcal{C}, \rightarrow)$:

$$\begin{array}{ccc} \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} \\ f_1 \uparrow & & \uparrow g_2 \\ \mathbf{A} & \xrightarrow{f_2} & \mathbf{B}_2. \end{array}$$

Since \mathcal{C} is closed with respect to finite products, \mathbf{A}^n , \mathbf{B}_1^n , \mathbf{B}_2^n are in \mathcal{C} . Since \mathcal{C} has the strict amalgamation property, there exists $\widehat{\mathbf{C}} \in \mathcal{C}$, $\hat{g}_1: \mathbf{B}_1^n \hookrightarrow \widehat{\mathbf{C}}$, $\hat{g}_2: \mathbf{B}_2^n \hookrightarrow \widehat{\mathbf{C}}$ such that the following is a pushout-square in $(\mathcal{C}, \rightarrow)$:

$$\begin{array}{ccc} \mathbf{B}_1^n & \xrightarrow{\hat{g}_1} & \widehat{\mathbf{C}} \\ f_1^n \uparrow & & \uparrow \hat{g}_2 \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n. \end{array}$$

Hence, there exists $k: \widehat{\mathbf{C}} \rightarrow \mathbf{C}^n$ such that the following diagram commutes:

$$(20) \quad \begin{array}{ccccc} & & g_1^n & \searrow & \mathbf{C}^n \\ & & \curvearrowright & & \uparrow \\ \mathbf{B}_1^n & \xrightarrow{\hat{g}_1} & \widehat{\mathbf{C}} & \xrightarrow{k} & \mathbf{C}^n \\ f_1^n \uparrow & & \uparrow \hat{g}_2 & & \uparrow g_2^n \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n & & \end{array}$$

Moreover, by Lemma 4.29, k is an embedding.

Next we note that there exists $h: \widehat{\mathbf{C}} \rightarrow \mathbf{T}$ such that the following diagram commutes:

$$(21) \quad \begin{array}{ccccc} & & h_1 & \searrow & \mathbf{T} \\ & & \curvearrowright & & \uparrow \\ \mathbf{B}_1^n & \xrightarrow{\hat{g}_1} & \widehat{\mathbf{C}} & \xrightarrow{h} & \mathbf{T} \\ f_1^n \uparrow & & \uparrow \hat{g}_2 & & \uparrow h_2 \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n & & \end{array}$$

Since \mathcal{C} has the HAP, there exist $\hat{k}: \mathbf{T} \hookrightarrow \mathbf{T}'$, and a homomorphism $\hat{h}: \mathbf{C}^n \rightarrow \mathbf{T}'$ such that the following diagram commutes:

$$(22) \quad \begin{array}{ccc} \mathbf{C}^n & \xrightarrow{\hat{h}} & \mathbf{T}' \\ \uparrow k & & \uparrow \hat{k} \\ \mathbf{C} & \xrightarrow{h} & \mathbf{T}. \end{array}$$

It remains to observe that the following diagram commutes:

$$(23) \quad \begin{array}{ccccc} & & & & \mathbf{T}' \\ & & & \nearrow \hat{h} & \\ & & & \hat{k} & \\ & & \mathbf{T} & & \\ & \nearrow h_1 & & & \\ \mathbf{B}_1^n & \xrightarrow{g_1^n} & \mathbf{C}^n & & \\ \uparrow f_1^n & & \uparrow g_2^n & & \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n & \xrightarrow{h_2} & \mathbf{T} \end{array}$$

Indeed, we compute:

$$\hat{k} \circ h_1 \stackrel{(21)}{=} \hat{k} \circ h \circ \hat{g}_1 \stackrel{(22)}{=} \hat{h} \circ k \circ \hat{g}_1 \stackrel{(20)}{=} \hat{h} \circ g_1^n.$$

Analogously the identity $\hat{k} \circ h_2 = \hat{h} \circ g_2^n$ may be shown. From these two identities it follows that diagram (23) commutes. Hence \mathcal{C} has the AEPⁿ, for every $n \in \mathbb{N} \setminus \{0\}$. \square

5. STRUCTURES WITH UNIVERSAL HOMOGENEOUS POLYMORPHISMS

5.1. Free homogeneous structures. Let $\underline{\Sigma}$ be a relational signature and let $\mathcal{C}_{\underline{\Sigma}}$ be the category of all $\underline{\Sigma}$ -structures with homomorphisms as morphisms.

Lemma 5.1. *Let $n \in \mathbb{N} \setminus \{0\}$, and for each $i \in \{1, 2\}$, let $\mathbf{A}, \mathbf{B}_i, \mathbf{C} \in \mathcal{C}_{\underline{\Sigma}}$, $f_i: \mathbf{A} \hookrightarrow \mathbf{B}_i$, $g_i: \mathbf{B}_i \hookrightarrow \mathbf{C}$, such that the following is a pushout-square in $\mathcal{C}_{\underline{\Sigma}}$:*

$$\begin{array}{ccc} \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} \\ \uparrow f_1 & & \uparrow g_2 \\ \mathbf{A} & \xrightarrow{f_2} & \mathbf{B}_2. \end{array}$$

Then the following is a weak pushout-square in $\mathcal{C}_{\underline{\Sigma}}$:

$$\begin{array}{ccc} \mathbf{B}_1^n & \xrightarrow{g_1^n} & \mathbf{C}^n \\ \uparrow f_1^n & & \uparrow g_2^n \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n. \end{array}$$

Proof. Let $\widehat{\mathbf{C}} \in \mathcal{C}_{\Sigma}$, $\hat{g}_i: \mathbf{B}_i^n \hookrightarrow \widehat{\mathbf{C}}$ (for $i \in \{1, 2\}$), such that the following is a pushout-square in \mathcal{C}_{Σ} .

$$\begin{array}{ccc} \mathbf{B}_1^n & \xrightarrow{\hat{g}_1} & \widehat{\mathbf{C}} \\ f_1^n \uparrow & & \uparrow \hat{g}_2 \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n. \end{array}$$

It remains to construct a homomorphism $h: \mathbf{C}^n \rightarrow \widehat{\mathbf{C}}$ such that the following diagram commutes:

$$(24) \quad \begin{array}{ccc} & \hat{g}_1 & \nearrow \\ \mathbf{B}_1^n & \xrightarrow{g_1^n} & \mathbf{C}^n & \xrightarrow{h} & \widehat{\mathbf{C}} \\ f_1^n \uparrow & & \uparrow g_2^n & & \uparrow \hat{g}_2 \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n & & \end{array}$$

We define

$$h(x_1, \dots, x_n) := \begin{cases} \hat{g}_1(u_1, \dots, u_n) & (x_1, \dots, x_n) = (g_1(u_1), \dots, g_1(u_n)) \\ \hat{g}_2(v_1, \dots, v_n) & (x_1, \dots, x_n) = (g_2(v_1), \dots, g_2(v_n)) \\ \hat{g}_1(u_1, \dots, u_1) & \text{else, if } g_1(u_1) = x_1 \\ \hat{g}_2(v_1, \dots, v_1) & \text{else, if } g_2(v_1) = x_1. \end{cases}$$

It remains to show that h is well-defined and a homomorphism. Suppose, that

$$(g_2(v_1), \dots, g_2(v_n)) = (x_1, \dots, x_n) = (g_1(u_1), \dots, g_1(u_n))$$

Since \mathbf{C} is the free amalgamated sum of $g_1(\mathbf{B}_1)$ with $g_2(\mathbf{B}_2)$ with respect to $g_1(f_1(\mathbf{A}))$, there exist $(a_1, \dots, a_n) \in \mathbf{A}^n$, such that $(f_1(a_1), \dots, f_1(a_n)) = (u_1, \dots, u_n)$ and $(f_2(a_1), \dots, f_2(a_n)) = (v_1, \dots, v_n)$. But since $\hat{g}_1 \circ f_1^n = \hat{g}_2 \circ f_2^n$, we obtain

$$\hat{g}_1(u_1, \dots, u_n) = \hat{g}_1(f_1(a_1), \dots, f_1(a_n)) = \hat{g}_2(f_2(a_1), \dots, f_2(a_n)) = \hat{g}_2(v_1, \dots, v_n).$$

If neither

$$(x_1, \dots, x_n) = (g_1(u_1), \dots, g_1(u_n)) \text{ nor } (x_1, \dots, x_n) = (g_2(v_1), \dots, g_2(v_n)),$$

but $g_1(u_1) = x_1 = g_2(v_1)$, then, since $g_1(\mathbf{B}_1) \cap g_2(\mathbf{B}_2) = g_1(f_1(\mathbf{A})) = g_2(f_2(\mathbf{A}))$, there exists $a_1 \in \mathbf{A}$ such that $f_1(a_1) = u_1$, and $f_2(a_1) = v_1$. Hence,

$$\hat{g}_1(u_1, \dots, u_1) = \hat{g}_1(f_1(a_1), \dots, f_1(a_1)) = \hat{g}_2(f_2(a_1), \dots, f_2(a_1)) = \hat{g}_2(v_1, \dots, v_1).$$

Thus, h is well-defined.

Let ϱ be a relational symbol of arity m from Σ , and let $(\bar{a}_1, \dots, \bar{a}_m) \in \varrho^{\mathbf{C}^n}$, where

$$\bar{a}_i = (a_{i,1}, \dots, a_{i,n}) \text{ (for } i \in \{1, \dots, m\}).$$

Then we have that $(a_{1,j}, \dots, a_{m,j})$ is in $\varrho^{\mathbf{C}}$, for each $j \in \{1, \dots, n\}$. Since $\varrho^{\mathbf{C}} = g_1(\varrho^{\mathbf{B}_1}) \cup g_2(\varrho^{\mathbf{B}_2})$, for every $j \in \{1, \dots, n\}$ we have $(a_{1,j}, \dots, a_{m,j}) \in g_1(\varrho^{\mathbf{B}_1})$ or $(a_{1,j}, \dots, a_{m,j}) \in g_2(\varrho^{\mathbf{B}_2})$.

Suppose that for every $j \in \{1, \dots, n\}$ there exists $(u_{1,j}, \dots, u_{m,j}) \in \varrho^{\mathbf{B}_1}$, such that

$$(a_{1,j}, \dots, a_{m,j}) = (g_1(u_{1,j}), \dots, g_1(u_{m,j})),$$

then we have

$$\begin{bmatrix} h(a_{1,1}, \dots, a_{1,n}) \\ \vdots \\ h(a_{m,1}, \dots, a_{m,n}) \end{bmatrix} = \begin{bmatrix} \hat{g}_1(u_{1,1}, \dots, u_{1,n}) \\ \vdots \\ \hat{g}_1(u_{m,1}, \dots, u_{m,n}) \end{bmatrix} \in \varrho^{\hat{\mathbf{C}}},$$

since \hat{g}_1 is a homomorphism.

Analogously, if for every $j \in \{1, \dots, n\}$ there exists $(v_{1,j}, \dots, v_{m,j}) \in \varrho^{\mathbf{B}_2}$, such that

$$(a_{1,j}, \dots, a_{m,j}) = (g_2(v_{1,j}), \dots, g_2(v_{m,j})),$$

then we have

$$\begin{bmatrix} h(a_{1,1}, \dots, a_{1,n}) \\ \vdots \\ h(a_{m,1}, \dots, a_{m,n}) \end{bmatrix} = \begin{bmatrix} \hat{g}_2(v_{1,1}, \dots, v_{1,n}) \\ \vdots \\ \hat{g}_2(v_{m,1}, \dots, v_{m,n}) \end{bmatrix} \in \varrho^{\hat{\mathbf{C}}},$$

since \hat{g}_2 is a homomorphism.

Otherwise, if there exists $(u_1, \dots, u_m) \in \varrho^{\mathbf{B}_1}$, such that

$$(a_{1,1}, \dots, a_{m,1}) = (g_1(u_1), \dots, g_1(u_m)),$$

then

$$\begin{bmatrix} h(a_{1,1}, \dots, a_{1,n}) \\ \vdots \\ h(a_{m,1}, \dots, a_{m,n}) \end{bmatrix} = \begin{bmatrix} \hat{g}_2(u_1, \dots, u_1) \\ \vdots \\ \hat{g}_2(u_m, \dots, u_m) \end{bmatrix} \in \varrho^{\hat{\mathbf{C}}},$$

and if there exists $(v_1, \dots, v_m) \in \varrho^{\mathbf{B}_2}$, such that $(a_{1,1}, \dots, a_{m,1}) = (g_2(v_1), \dots, g_2(v_m))$, then

$$\begin{bmatrix} h(a_{1,1}, \dots, a_{1,n}) \\ \vdots \\ h(a_{m,1}, \dots, a_{m,n}) \end{bmatrix} = \begin{bmatrix} \hat{g}_2(v_1, \dots, v_1) \\ \vdots \\ \hat{g}_2(v_m, \dots, v_m) \end{bmatrix} \in \varrho^{\hat{\mathbf{C}}}.$$

Thus, h is a homomorphism.

By construction of h we have that diagram (24) commutes. Thus, the proof is complete. \square

Proposition 5.2. *Let \mathbf{U} be a countably infinite homogeneous relational structure whose age has the free amalgamation property. Then $\text{Age}(\mathbf{U})$ has the AEPⁿ, for every $n \in \mathbb{N} \setminus \{0\}$.*

Proof. Let $n \in \mathbb{N} \setminus \{0\}$.

Given $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{T} \in \text{Age}(\mathbf{U})$, $f_1: \mathbf{A} \hookrightarrow \mathbf{B}_1$, $f_2: \mathbf{A} \hookrightarrow \mathbf{B}_2$, $h_1: \mathbf{B}_1 \rightarrow \mathbf{T}$, $h_2: \mathbf{B}_2 \rightarrow \mathbf{T}$. such that $h_1 \circ f_1 = h_2 \circ f_2$. Without loss of generality, f_1 and f_2 are identical embeddings and $B_1 \cap B_2 = A$.

Let $\mathbf{C} := \mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$, in other words, the following is a pushout-square in \mathcal{C}_{Σ} :

$$\begin{array}{ccc} \mathbf{B}_1 & \xrightarrow{=} & \mathbf{C} \\ \uparrow = & & \uparrow = \\ \mathbf{A} & \xrightarrow{=} & \mathbf{B}_2. \end{array}$$

By Lemma 5.1, the following is a weak pushout square in \mathcal{C}_{Σ} :

$$\begin{array}{ccc} \mathbf{B}_1^n & \xrightarrow{=} & \mathbf{C}^n \\ \uparrow = & & \uparrow = \\ \mathbf{A}^n & \xrightarrow{=} & \mathbf{B}_2^n. \end{array}$$

Hence there exists some $h : \mathbf{C}^n \rightarrow \mathbf{T}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & & \mathbf{T} \\
 & \nearrow^{h_1} & \nearrow^h \\
 \mathbf{B}_1^n & \xrightarrow{=} & \mathbf{C}^n \\
 \uparrow = & & \uparrow = \\
 \mathbf{A}^n & \xrightarrow{=} & \mathbf{B}_2^n \\
 & \searrow_{h_2} & \searrow
 \end{array}$$

Taking $\mathbf{T}' := \mathbf{T}$, we obtain, that the following diagram commutes, too:

$$\begin{array}{ccc}
 & & \mathbf{T}' \\
 & \nearrow^h & \nearrow^= \\
 & \nearrow^{h_1} & \nearrow \\
 \mathbf{B}_1^n & \xrightarrow{=} & \mathbf{C}^n \\
 \uparrow = & & \uparrow = \\
 \mathbf{A}^n & \xrightarrow{=} & \mathbf{B}_2^n \\
 & \searrow_{h_2} & \searrow
 \end{array}$$

Thus, $\text{Age}(\mathbf{U})$ has the AEPⁿ. □

Corollary 5.3. *Let \mathbf{U} be a countably infinite homogeneous relational structure whose age has the free amalgamation property. Let $n \in \mathbb{N} \setminus \{0\}$. Then \mathbf{U} has an n -ary universal homogeneous polymorphism if and only if $\text{Age}(\mathbf{U})$ has the HAPⁿ.*

Proof. This follows directly from Proposition 5.2, in conjunction with Theorem 4.26. □

Example 5.4. The Rado-graph has universal homogeneous polymorphisms of every arity, since its age is closed with respect to finite products, has the HAP, and has the free amalgamation property.

For the same reasons, the countable universal homogeneous digraph and the countable universal homogeneous k -hypergraphs have universal homogeneous polymorphisms of all arities.

5.2. The generic poset with \leq . The following construction of amalgamated free sums in the category of posets is folklore:

Construction. Let \mathbf{A} , \mathbf{B}_1 , \mathbf{B}_2 be posets such that $\mathbf{A} \leq \mathbf{B}_1$, $\mathbf{A} \leq \mathbf{B}_2$, and such that $B_1 \cap B_2 = A$. Define $C := B_1 \cup B_2$,

$$(\leq_C) := (\leq_{\mathbf{B}_1}) \cup (\leq_{\mathbf{B}_2}) \cup \sigma \cup \tau,$$

where

$$\begin{aligned}
 \sigma &= \{(b_1, b_2) \mid b_1 \in B_1, b_2 \in B_2, \exists a \in A : b_1 \leq_{\mathbf{B}_1} a \leq_{\mathbf{B}_2} b_2\}, \\
 \tau &= \{(b_2, b_1) \mid b_1 \in B_1, b_2 \in B_2, \exists a \in A : b_2 \leq_{\mathbf{B}_2} a \leq_{\mathbf{B}_1} b_1\},
 \end{aligned}$$

and finally $\mathbf{C} := (C, \leq_{\mathbf{C}})$. Then $\mathbf{C} = \mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$. In particular, the following is a pushout-square in the category of posets:

$$\begin{array}{ccc} \mathbf{B}_1 & \xhookrightarrow{\quad} & \mathbf{C} \\ \uparrow \scriptstyle = & & \uparrow \scriptstyle = \\ \mathbf{A} & \xhookrightarrow{\quad} & \mathbf{B}_2. \end{array}$$

Lemma 5.5. *The class of finite posets has well-behaved amalgamated free sums.*

Proof. Given finite posets $\mathbf{A}_1, \mathbf{B}_{1,1}, \mathbf{B}_{1,2}, \mathbf{A}_2, \mathbf{B}_{2,1}, \mathbf{B}_{2,2}$, such that $\mathbf{A}_1 \leq \mathbf{B}_{1,1}, \mathbf{A}_1 \leq \mathbf{B}_{1,2}, B_{1,1} \cap B_{1,2} = A_1, \mathbf{A}_2 \leq \mathbf{B}_{2,1}, \mathbf{A}_2 \leq \mathbf{B}_{2,2}, B_{2,1} \cap B_{2,2} = A_2$.

Let $\mathbf{C}_1 := \mathbf{B}_{1,1} \oplus_{\mathbf{A}_1} \mathbf{B}_{1,2}, \mathbf{C}_2 := \mathbf{B}_{2,1} \oplus_{\mathbf{A}_2} \mathbf{B}_{2,2}$, and let $\mathbf{D} := (\mathbf{B}_{1,1} \times \mathbf{B}_{2,1}) \oplus_{\mathbf{A}_1 \times \mathbf{A}_2} (\mathbf{B}_{1,2} \times \mathbf{B}_{2,2})$. We will show that $\mathbf{D} \leq \mathbf{C}_1 \times \mathbf{C}_2$.

First we note

$$\begin{aligned} D &= B_{1,1} \times B_{2,1} \cup B_{1,2} \times B_{2,2} \\ &\subseteq B_{1,1} \times B_{1,2} \cup B_{1,1} \times B_{2,2} \cup B_{1,2} \times B_{2,1} \cup B_{1,2} \times B_{2,2}, \\ &= C_1 \times C_2. \end{aligned}$$

Now we will show that $(\leq_{\mathbf{D}}) = (\leq_{\mathbf{C}_1 \times \mathbf{C}_2}) \cap D^2$.

“ \subseteq .” Let $(u_1, u_2), (v_1, v_2) \in D$, such that $(u_1, u_2) \leq_{\mathbf{D}} (v_1, v_2)$. If $(u_1, u_2), (v_1, v_2) \in B_{1,1} \times B_{2,1}$, then

$$\begin{aligned} (u_1, u_2) \leq_{\mathbf{D}} (v_1, v_2) &\iff (u_1, u_2) \leq_{\mathbf{B}_{1,1} \times \mathbf{B}_{2,1}} (v_1, v_2) \\ &\iff u_1 \leq_{\mathbf{B}_{1,1}} v_1 \wedge u_2 \leq_{\mathbf{B}_{2,1}} v_2 \\ &\iff u_1 \leq_{\mathbf{C}_1} v_1 \wedge u_2 \leq_{\mathbf{C}_2} v_2 \\ &\iff (u_1, u_2) \leq_{\mathbf{C}_1 \times \mathbf{C}_2} (v_1, v_2). \end{aligned}$$

Analogously, if $(u_1, u_2), (v_1, v_2) \in B_{1,2} \times B_{2,2}$, then $(u_1, u_2) \leq_{\mathbf{D}} (v_1, v_2)$ if and only if $(u_1, u_2) \leq_{\mathbf{C}_1 \times \mathbf{C}_2} (v_1, v_2)$.

Suppose that $(u_1, u_2) \in B_{1,1} \times B_{2,1}, (v_1, v_2) \in B_{1,2} \times B_{2,2}$. Then

$$\begin{aligned} (u_1, u_2) \leq_{\mathbf{D}} (v_1, v_2) &\iff \exists (a_1, a_2) \in A_1 \times A_2 : \\ &\quad (u_1, v_1) \leq_{\mathbf{B}_{1,1} \times \mathbf{B}_{2,1}} (a_1, a_2) \leq_{\mathbf{B}_{1,2} \times \mathbf{B}_{2,2}} (v_1, v_2) \\ &\iff \exists (a_1, a_2) \in A_1 \times A_2 : \\ &\quad u_1 \leq_{\mathbf{B}_{1,1}} a_1 \leq_{\mathbf{B}_{1,2}} v_1 \wedge u_2 \leq_{\mathbf{B}_{2,1}} a_2 \leq_{\mathbf{B}_{2,2}} v_2 \\ &\iff u_1 \leq_{\mathbf{C}_1} v_1 \wedge u_2 \leq_{\mathbf{C}_2} v_2 \\ &\iff (u_1, u_2) \leq_{\mathbf{C}_1 \times \mathbf{C}_2} (v_1, v_2). \end{aligned}$$

Analogously the case $(u_1, u_2) \in B_{1,2} \times B_{2,2}, (v_1, v_2) \in B_{1,1} \times B_{2,1}$ is handled. \square

Corollary 5.6. *The class of finite posets has the AEPⁿ.*

Proof. This follows directly from Lemma 5.5, in conjunction with Proposition 4.30. \square

Lemma 5.7 ([13, Example 3.4]). *The class of finite posets has the HAP.*

Corollary 5.8. *The class of finite posets has the HAPⁿ, for every $n \in \mathbb{N} \setminus \{0\}$.*

Proof. This follows directly from the fact that the class of finite posets is closed under finite products and has the HAP (cf. Lemma 5.7). \square

Theorem 5.9. *The generic poset has universal homogeneous polymorphisms of every arity.*

Proof. Let (\mathbb{P}, \leq) be the countable generic poset. The age \mathcal{C} of (\mathbb{P}, \leq) consists of all finite posets. By Lemma 5.6, we have that \mathcal{C} has the AEPⁿ, for every $n \in \mathbb{N} \setminus \{0\}$. By Lemma 5.8, \mathcal{C} has the HAPⁿ, for every $n \in \mathbb{N} \setminus \{0\}$. Finally, by Theorem 4.26, \mathbb{P} has universal homogeneous polymorphisms of every arity. \square

6. STRUCTURES WITHOUT UNIVERSAL HOMOGENEOUS POLYMORPHISMS

6.1. The rational Urysohn-space. Consider the relational signature Σ_M that contains for every $r \in \mathbb{Q}_0^+$ a binary relational symbol ϱ_r (here and below, by \mathbb{Q}^+ we will denote the set of positive rationals, and by \mathbb{Q}_0^+ we will denote the set of non-negative rationals). Then to every metric space (A, d) we may associate a Σ_M -structures \mathbf{A} by defining

$$\varrho_r^{\mathbf{A}} := \{(x, y) \in A^2 \mid d(x, y) < r\},$$

for every $r \in \mathbb{Q}^+ \cup \{0\}$. The metric d can be reconstructed from \mathbf{A} by

$$d(x, y) = \inf\{r \in \mathbb{Q}_0^+ \mid (x, y) \in \varrho_r^{\mathbf{A}}\}.$$

To make this correspondence functorial, the proper choice of morphisms between metric spaces are the non-expansive maps. Recall that a function $f: (A, d_A) \rightarrow (B, d_B)$ is called *non-expansive* if for all $x, y \in A$ we have

$$d_B(f(x), f(y)) \leq d_A(x, y).$$

With this definition of morphisms between metric spaces, the assignment $R: (A, d) \mapsto \mathbf{A}$, $R: f \mapsto f$ is a full embedding into the category \mathcal{C}_{Σ_M} of all Σ_M -structures with homomorphisms as morphisms. Therefore, in the following we will identify metric spaces with their relational counter-parts.

The direct product of two metric spaces is constructed as follows:

Construction. Let $\mathbf{A} = (A, d_A)$ and $\mathbf{B} = (B, d_B)$ be metric spaces. On $A \times B$ we define a metric d as follows:

$$d((x, y), (u, v)) := \max(d_A(x, u), d_B(y, v))$$

Then $\mathbf{A} \times \mathbf{B} := (A \times B, d)$ is called the *direct product* of \mathbf{A} and \mathbf{B}

Remark. $\mathbf{A} \times \mathbf{B}$ is actually *the* product in the category of metric spaces with non-expansive maps. The functor R that assigns a Σ_M -structure to each metric space preserves products.

In the category of non-empty metric spaces with non-expansive maps the amalgamated free sum is constructed as follows:

Construction. Let $\mathbf{A} = (A, d_A)$, $\mathbf{B}_1 = (B_1, d_{B_1})$, $\mathbf{B}_2 = (B_2, d_{B_2})$ be metric spaces such that $\emptyset \neq A = B_1 \cap B_2$ and such that the identical embeddings from A to B_1 and B_2 are both isometries. Then on $B_1 \cup B_2$ we define a metric d according to:

$$d(x, y) := \begin{cases} d_{B_1}(x, y) & x, y \in B_1, \\ d_{B_2}(x, y) & x, y \in B_2, \\ \inf\{d_{B_1}(x, z) + d_{B_2}(z, y) \mid z \in A\} & x \in B_1, y \in B_2, \\ \inf\{d_{B_2}(x, z) + d_{B_1}(z, y) \mid z \in A\} & x \in B_2, y \in B_1. \end{cases}$$

The metric space $(B_1 \cup B_2, d)$ is denoted by $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$ and is called the amalgamated free sum of \mathbf{B}_1 and \mathbf{B}_2 with respect to \mathbf{A} .

The construction of amalgamated free sums of metric spaces may be rephrased in the language of $\underline{\Sigma}_M$ -structures as follows (to shorten the notations we define $\mathbf{D} := \mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$):

$$\varrho_r^{\mathbf{D}} = \varrho_r^{\mathbf{B}_1} \cup \varrho_r^{\mathbf{B}_2} \cup \sigma_r \cup \tau_r,$$

where

$$\sigma_r = \{(x, y) \in B_1 \times B_2 \mid \exists s, t \in \mathbb{Q}_0^+, z \in A : s + t \leq r \wedge (x, z) \in \varrho_s^{\mathbf{B}_1} \wedge (z, y) \in \varrho_t^{\mathbf{B}_2}\},$$

$$\tau_r = \{(x, y) \in B_2 \times B_1 \mid \exists s, t \in \mathbb{Q}_0^+, z \in A : s + t \leq r \wedge (x, u) \in \varrho_s^{\mathbf{B}_2} \wedge (u, y) \in \varrho_t^{\mathbf{B}_1}\},$$

Remark. The functor R that assigns a $\underline{\Sigma}_M$ -structure to every metric space does *not* preserve amalgamated free sums. However, R *will* preserve amalgamated free sums if we restrict the attention to the quasi-variety of $\underline{\Sigma}_M$ -structures defined by the quasi-identities:

$$\begin{aligned} \varrho_0(x, y) &\Rightarrow x = y, \\ x = y &\Rightarrow \varrho_0(x, y), \\ \varrho_i(x, y) &\Rightarrow \varrho_i(y, x) \quad (\forall i \in \mathbb{Q}_0^+), \\ \varrho_i(x, y) &\Rightarrow \varrho_j(x, y) \quad (\forall i, j \in \mathbb{Q}_0^+ \text{ with } i < j), \\ \varrho_i(x, y) \wedge \varrho_j(y, z) &\Rightarrow \varrho_{i+j}(x, z) \quad (\forall i, j \in \mathbb{Q}_0^+). \end{aligned}$$

Proposition 6.1. *The class of finite metric spaces does not have the AEPⁿ for any $n > 1$.*

Proof. Let $n > 1$. Suppose on the contrary that the class finite metric spaces has the AEPⁿ.

Consider the metric space $\mathbf{B}_1 = (B_1, d_1)$ where $B_1 = \{a, u\}$ and where $d_1(a, u) = 1$, and $\mathbf{B}_2 = (B_2, d_2)$ where $B_2 = \{a, v\}$ and where $d_2(a, v) = 10$. Let \mathbf{A} be the joint metric subspace of \mathbf{B}_1 and \mathbf{B}_2 that consists just of one point a .

Let $\mathbf{T} := \mathbf{B}_1^n \oplus_{\mathbf{A}^n} \mathbf{B}_2^n$, and let $h_1 : \mathbf{B}_1 \hookrightarrow \mathbf{T}$, $h_2 : \mathbf{B}_2 \hookrightarrow \mathbf{T}$ be the identical embeddings.

By the AEPⁿ, there exists a finite metric space \mathbf{C} , \mathbf{T}' , embeddings $g_1 : \mathbf{B}_1 \hookrightarrow \mathbf{C}$ and $g_2 : \mathbf{B}_2 \hookrightarrow \mathbf{C}$, a nonexpansive map $h : \mathbf{C}^n \rightarrow \mathbf{T}'$, and an embedding $k : \mathbf{T} \hookrightarrow \mathbf{T}'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & & & \mathbf{T}' \\ & & & \nearrow h & \\ & & \mathbf{T} & \nearrow k & \\ & \nearrow h_1 & & & \\ \mathbf{B}_1^n & \xrightarrow{g_1^n} & \mathbf{C}^n & & \\ \uparrow f_1^n & & \uparrow g_2^n & \nearrow h_2 & \\ \mathbf{A}^n & \xrightarrow{f_2^n} & \mathbf{B}_2^n & & \end{array}$$

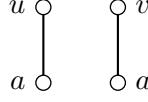
and such that $g_1 \circ f_1 = g_2 \circ f_2$. Here, without loss of generality, k, g_1, g_2 are identical embeddings. Consider the elements $\bar{x} := (a, u, \dots, u) \in \mathbf{B}_1^n$, $\bar{y} := (v, a, \dots, a) \in \mathbf{B}_2^n$, $\bar{a} = (a, \dots, a) \in \mathbf{A}^n$. Then we have

$$d_{\mathbf{C}^n}(\bar{x}, \bar{y}) \leq \max\{d_{\mathbf{B}_1}(u, a), d_{\mathbf{B}_2}(a, v)\} = 10$$

However,

$$d_{\mathbf{T}'}(\bar{x}, \bar{y}) = d_{\mathbf{T}}(\bar{x}, \bar{y}) = d_{\mathbf{B}_1^n}(\bar{x}, \bar{a}) + d_{\mathbf{B}_2^n}(\bar{a}, \bar{y}) = 1 + 10 = 11.$$

However, this shows that h is not nonexpansive—a contradiction. \square

FIGURE 1. The chains \mathbf{B}_1 and \mathbf{B}_2

Corollary 6.2. *The rational Urysohn space does not have universal homogeneous polymorphisms of any arity $n > 1$.*

Remark. Note that the rational Urysohn space *does* have universal homogeneous endomorphisms (cf. [33, Example 4.10]).

6.2. The rationals with \leq .

Proposition 6.3. *The class of finite chains (with reflexive order relations) does not have the AEP n , for every $n \geq 2$.*

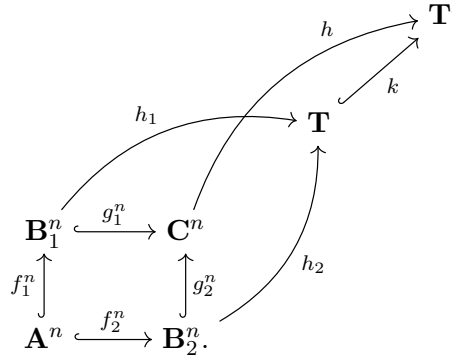
Proof. Suppose that the class of finite chains has the AEP n . Let the chains $\mathbf{B}_1 = (\{a, u\}, \leq_1)$ and $\mathbf{B}_2 = (\{a, v\}, \leq_2)$ be given like in Figure 1. Let \mathbf{A} be the trivial chain that consists only of the element a . Let $f_1: \mathbf{A} \hookrightarrow \mathbf{B}_1$ and $f_2: \mathbf{A} \hookrightarrow \mathbf{B}_2$ be the identical embeddings. Consider the chain $\mathbf{T} = (T, \leq_{\mathbf{T}})$ given by $T = \{aa, au, av, va, vv, ua, uu\}$ and

$$aa <_{\mathbf{T}} au <_{\mathbf{T}} av <_{\mathbf{T}} va <_{\mathbf{T}} vv <_{\mathbf{T}} ua <_{\mathbf{T}} uu.$$

Consider the homomorphisms $h_i: \mathbf{B}_i^n \rightarrow \mathbf{T}$ ($i \in \{1, 2\}$), given through

$$h_i: (x_1, \dots, x_n) \mapsto x_1 x_2.$$

Then, by assumption, there exist finite chains \mathbf{C} , \mathbf{T}' , embeddings $g_1: \mathbf{B}_1 \hookrightarrow \mathbf{C}$, $g_2: \mathbf{B}_2 \hookrightarrow \mathbf{C}$, $k: \mathbf{T} \hookrightarrow \mathbf{T}'$, as well as a homomorphism $h: \mathbf{C}^n \rightarrow \mathbf{T}'$, such that the following diagram commutes:



Without loss of generality, we may assume that g_1 , g_2 , and k are identical embeddings. In \mathbf{C} either of the following inequalities hold:

$$u < v, \quad u = v, \quad u > v.$$

Suppose that $u < v$ in \mathbf{C} . Consider $\bar{u} = (u, u, \dots, u) \in \mathbf{B}_1^n$ and $\bar{v} = (v, v, \dots, v) \in \mathbf{B}_2^n$. Then $k(h_1(\bar{u})) = uu > vv = k(h_2(\bar{v}))$, while $g_1^n(\bar{u}) < g_2^n(\bar{v})$. However, then $k(h_1(\bar{u})) = h(g_1^n(\bar{u})) < h(g_2^n(\bar{v})) = k(h_2(\bar{v}))$ —a contradiction.

Suppose that $u = v$ in \mathbf{C} . Consider again $\bar{u} = (u, u, \dots, u) \in \mathbf{B}_1^n$ and $\bar{v} = (v, v, \dots, v) \in \mathbf{B}_2^n$. Then $k(h_1(\bar{u})) = uu \neq vv = k(h_2(\bar{v}))$. On the other hand we have $g_1^n(\bar{u}) = g_2^n(\bar{v})$, and thus $k(h_1(\bar{u})) = h(g_1^n(\bar{u})) = h(g_2^n(\bar{v})) = k(h_2(\bar{v}))$ —a contradiction.

Finally suppose that $u > v$ in \mathbf{C} . Consider $\bar{u} = (a, u, u, \dots, u) \in \mathbf{B}_1^n$ and $\bar{v} = (a, v, v, \dots, v) \in \mathbf{B}_2^n$. Then $k(h_1(\bar{u})) = au < av = k(h_2(\bar{v}))$, and $g_1^n(\bar{u}) > g_2^n(\bar{v})$. Hence $k(h_1(\bar{u})) = h(g_1^n(\bar{u})) > h(g_2^n(\bar{v})) = k(h_2(\bar{v}))$ —a contradiction.

It follows that the class of finite chains does not have the AEPⁿ, for all $n \geq 2$. \square

Corollary 6.4. *The structure (\mathbb{Q}, \leq) does not have universal homogeneous polymorphisms of any arity $n > 1$.*

7. CLONES WITH AUTOMATIC HOMEOMORPHICITY

Theorem 7.1. *Let \mathbf{U} be a countable homogeneous relational structure such that*

- (1) $\text{Pol}(\mathbf{U})$ contains all constant functions,
- (2) $\text{Age}(\mathbf{U})$ has the free amalgamation property,
- (3) $\text{Age}(\mathbf{U})$ is closed with respect to finite products,
- (4) $\text{Age}(\mathbf{U})$ has the HAP.

Then $\text{Pol}(\mathbf{U})$ has automatic homeomorphicity.

Proof. Let h be an isomorphism of $\text{Pol}(\mathbf{U})$ to the polymorphism clone of another countable structure. Since $\text{Pol}(\mathbf{U})$ contains all constant functions, it follows from Proposition 3.12, that h is open. Since $\text{Age}(\mathbf{U})$ has the HAP, and is closed with respect to finite products, it follows that it has the HAPⁿ, for all $n \in \mathbb{N} \setminus \{0\}$. Thus, since $\text{Age}(\mathbf{U})$ has the free amalgamation property, it follows from Corollary 5.3, that \mathbf{U} has universal homogeneous polymorphisms of all arities. Thus, by Proposition 3.25, it follows that $\text{Pol}(\mathbf{U})$ has a strong gate covering. Since h is open, it follows from Proposition 3.6, that h is a homeomorphism. \square

Corollary 7.2. *The polymorphism clones of the following structures have automatic homeomorphicity:*

- the structure $(\mathbb{N}, =)$ (shown already in [6, Corollary 28]),
- the Rado graph with all loops added,
- the universal homogeneous digraph with all loops added,

Theorem 7.3. *The polymorphism clone of the generic poset (\mathbb{P}, \leq) has automatic homeomorphicity.*

Proof. Let h be an isomorphism from $\text{Pol}(\mathbb{P}, \leq)$ to the polymorphism clone of another countable structure.

Clearly, all constant functions are polymorphisms of (\mathbb{P}, \leq) . Thus, by Proposition 3.12, h is open.

By Theorem 5.9, (\mathbb{P}, \leq) has universal homogeneous polymorphisms of all arities. By Proposition 3.25, $\text{Pol}(\mathbb{P}, \leq)$ has a strong gate covering. Since h is open, by Proposition 3.6, h is a homeomorphism. \square

Theorem 7.4. *Let \mathbf{U} be a countable ω -categorical homogeneous relational structure and let \mathcal{K} be a set of structures on U , containing \mathbf{U} . Suppose that*

- (1) $\text{Aut}(\mathbf{U})$ acts transitively on U ,
- (2) $\bar{\text{Aut}}(\mathbf{U})$ has automatic homeomorphicity with respect to \mathcal{K} ,
- (3) $\text{Age}(\mathbf{U})$ has the free amalgamation property,
- (4) $\text{Age}(\mathbf{U})$ is closed with respect to finite products,
- (5) $\text{Age}(\mathbf{U})$ has the HAP.

Then $\text{Pol}(\mathbf{U})$ has automatic homeomorphicity with respect to \mathcal{K} .

Proof. Let h be an isomorphism from $\text{Pol}(\mathbf{U})$ to the polymorphism clone of a member of \mathcal{K} . Since $\text{Age}(\mathbf{U})$ has the HAP, and is closed with respect to finite products, it follows that it has the HAP^n , for all $n \in \mathbb{N} \setminus \{0\}$. Thus, since $\text{Age}(\mathbf{U})$ has the free amalgamation property, it follows from Corollary 5.3, that \mathbf{U} has universal homogeneous polymorphisms of all arities. Thus, by Proposition 3.25, it follows that $\text{Pol}(\mathbf{U})$ has a strong gate covering. Since $\overline{\text{Aut}}(\mathbf{U})$ has automatic homeomorphicity, it follows that the restriction of h to $\overline{\text{Aut}}(\mathbf{U})$ is a topological embedding. In particular, $h|_{\overline{\text{Aut}}(\mathbf{U})}$ is continuous. Consequently, since $\text{Pol}(\mathbf{U})$ has a strong gate covering, it follows from Lemma 3.5 that h is continuous, too. Thus, since $\text{Aut}(\mathbf{U})$ acts transitively on U , $\text{Age}(\mathbf{U})$ has the free amalgamation property, $\text{Age}(\mathbf{U})$ is closed with respect to finite products, and $\text{Age}(\mathbf{U})$ has the HAP, it follows from Proposition 3.16 that h is a homeomorphism. \square

Corollary 7.5. *The polymorphism clones of the following countably infinite structures have automatic homeomorphicity:*

- the Rado-graph (shown already in [6, Theorem 50]),
- the universal homogeneous digraph,
- the universal homogeneous k -uniform hypergraph (for all $k \geq 2$),

8. CONCLUDING REMARKS

Both of our strategies for proving automatic homeomorphicity of clones base on strong gate coverings. We obtained these gate coverings from universal homogeneous polymorphisms. The property of a structure to have universal homogeneous polymorphisms appears to be *much* stronger than the property of its polymorphism clone to have a strong gate covering. So, even though a structure may fail to have universal homogeneous polymorphisms, it could still have a strong gate covering.

In the course of preparing this paper we considered altogether 6 alternative definitions of universal homogeneous polymorphisms—each related with a different kind of comma-categories. We could show that these 6 definitions fall into two equivalence classes. Both alternatives are good for constructing strong gate coverings. However, none of them seemed to be more general than the other. So, for this paper we settled to consider the technically easiest definition.

Problem 1. Find more general criteria on the age of a structure that imply the existence of strong gate coverings for its polymorphism clone.

At some point while writing this paper we thought that we had proven the automatic homeomorphicity for the polymorphism clone of the rational Urysohn space. The reason for this believe was that almost all ingredients for using our first strategy seemed to be in place. Every constant function is a polymorphism of the rational Urysohn space. The class of finite rational metric spaces has the HAP^n . Moreover, finite rational metric spaces are closed under amalgamated free sums of *non-empty* metric spaces. The trouble was that we could not show that these amalgamated free sums are well-behaved. Eventually, we showed that the AEP^n fails for the class of finite rational metric spaces. Still we hope that our strategy may in future lead to a proof of automatic homeomorphicity of the rational Urysohn space.

Problem 2. Does the polymorphism clone of the rational Urysohn space have a strong gate covering (and hence automatic homeomorphicity)?

The situation is similar with (\mathbb{Q}, \leq) . So we ask:

Problem 3. Does the polymorphism clone of (\mathbb{Q}, \leq) have a strong gate covering (and hence automatic homeomorphicity)?

Our techniques appear to work very well for homogeneous structures whose age has the free amalgamation property. It is natural to ask then, how essential is the postulate of the free amalgamation property? From one hand we were able to apply our methods to the countable generic poset. This is an example of a structure whose age has the strict amalgamation property. Its age is also axiomatizable by quasi-identities. In general, whenever an age can be axiomatized by quasi-identities, then we have a nice description of amalgamated free sums. However, we could not find good condition on the quasi-identities, to assure that the amalgamated free sums are well-behaved. In fact, it turns out that there exist homogeneous structures, whose age has the strict amalgamation property and is axiomatizable by quasi-identities, but whose polymorphism clone does not have universal homogeneous polymorphisms of arity ≥ 2 . Consider, e.g., the class of finite rational partial metric spaces (these are metric spaces in which the distance $+\infty$ is allowed). This class is an age, it has the strict amalgamation property, and it is axiomatizable by quasi-identities. Yet, our example that shows that the class of finite rational metric spaces fails to have the AEPⁿ for every $n \geq 2$, works also for the class of finite rational partial metric spaces. So we pose the following problem:

Problem 4. Find natural conditions, under which the amalgamated free sums in strict Fraïssé classes are well-behaved. More concretely, find conditions under which the amalgamated free sum in a quasi-variety of relational structures is well-behaved.

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